Math 2210-3
Wednesday Jan 27 §11.5

HW for Wed. Feb. 3
§11.5 3, 9, 13 a b, 19, 32, 34, 39, 41, 43, 44
11.6 5, 9, 17, 19, 28, 29a, 23, 25, 30a
11.7 1, 4

- Finish Monday notes about parametric lines, with position vectors

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$
$$= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

i.e. the position vector (from the origin) ends at points $(x, y, z)$ with

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

as the parameter $t$ varies

Today we begin discussing general parametric curves, whose position vectors are given by vector-valued functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$$

position vector description

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

equivalent description

- often an interval is specified for the $t$ values, i.e. the domain of $\vec{r}(t)$: $a \le t \le b$

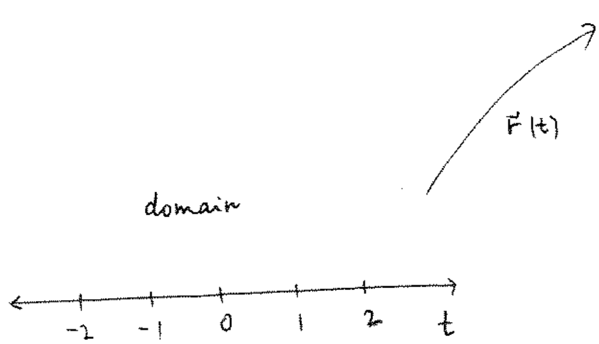
- if the curve lies in the $x$-$y$ plane $\vec{r}(t) = \langle f(t), g(t) \rangle$ i.e. $x = f(t)$, $y = g(t)$

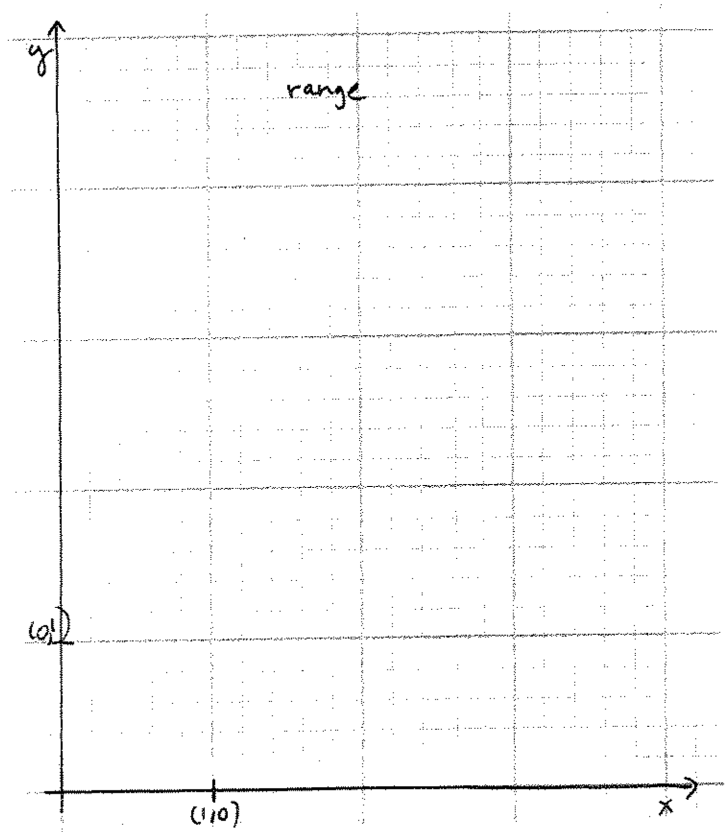
Example 1 $\vec{r}(t) = \langle 2+t, 1+t^2 \rangle$
$-2 \le t \le 2$

(a) plot some points on this curve

(b) Find the Cartesian (x-y) eqtn satisfied by all points on the curve, then sketch it ("Solve for t")

| $t$ | $\vec{r}(t)$ |
|---|---|
| -2 | |
| -1 | |
| 0 | |
| 1 | |
| 2 | |



Example 2

If the curve $x = f(t)$, $y = g(t)$ is locally a graph $y = F(x)$, then the slope of this graph is
near some given point

$$F'(x) = \frac{dy}{dx}$$

but $x, y$ are also fcns of $t$, so

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \quad \text{1210 chainrule!}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

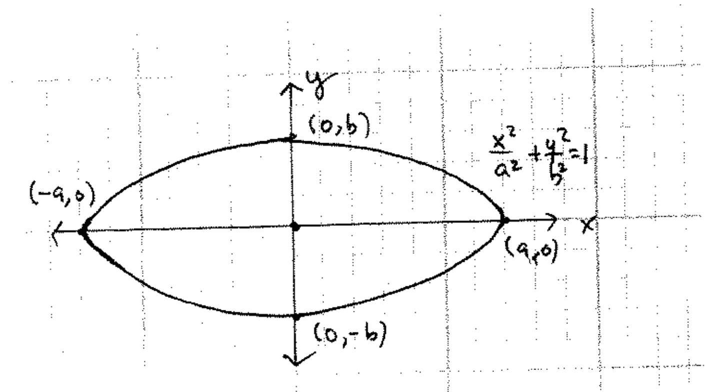
Compute $\frac{dy}{dx}$ both ways (see box) for Example 1, at the point $(1, 2)$

Some favorite curves in the plane:

Example 3 Show $\boxed{\vec{F}(t) = \langle a\cos t,\ b\sin t\rangle}$ $(a, b > 0)$

is the position vector of a curve traveling around the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



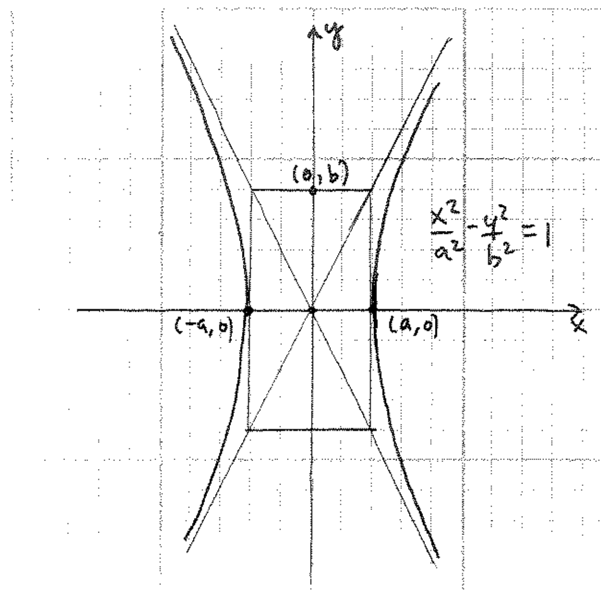

What is a good domain $t$-interval so that $\vec{F}(t)$ describes a simple closed curve, traveling once around the ellipse?

simple ← "$\vec{F}(t)$ doesn't cross itself" i.e. no self intersections, $\vec{F}(t)$ is one to one

closed ← starts & ends at the same place, $\vec{F}(a) = \vec{F}(b)$

Example 4 Show $F(t) = \langle a\cosh t, b\sinh t\rangle$
$(a, b > 0)$

parameterizes half of the hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$
How could you parameterize the other half?




Recall,
$$\cosh t = \tfrac{1}{2}(e^t + e^{-t})$$
$$\sinh t = \tfrac{1}{2}(e^t - e^{-t})$$
$$\cosh^2 t - \sinh^2 t = 1 \;!$$

Example 5: Sketch the curve of intersection between the cylinder
$$x^2 + y^2 = 4$$
and the plane $y + z = 2$
Then, find a parametric representation of this curve!

Calculus for parametric curves

parametric curves with position vector

$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \longleftrightarrow \quad$ in x-y plane

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad \longleftrightarrow \quad$ in $\mathbb{R}^3$

$$\boxed{\vec{r}\,'(t) := \lim_{\Delta t \to 0} \underbrace{\left(\tfrac{1}{\Delta t}\right)\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right)}} \quad \boxed{\text{definition.}}$$

We call $\vec{r}\,'(t)$ the <u>velocity vector</u> (if t is time) or <u>tangent vector</u>

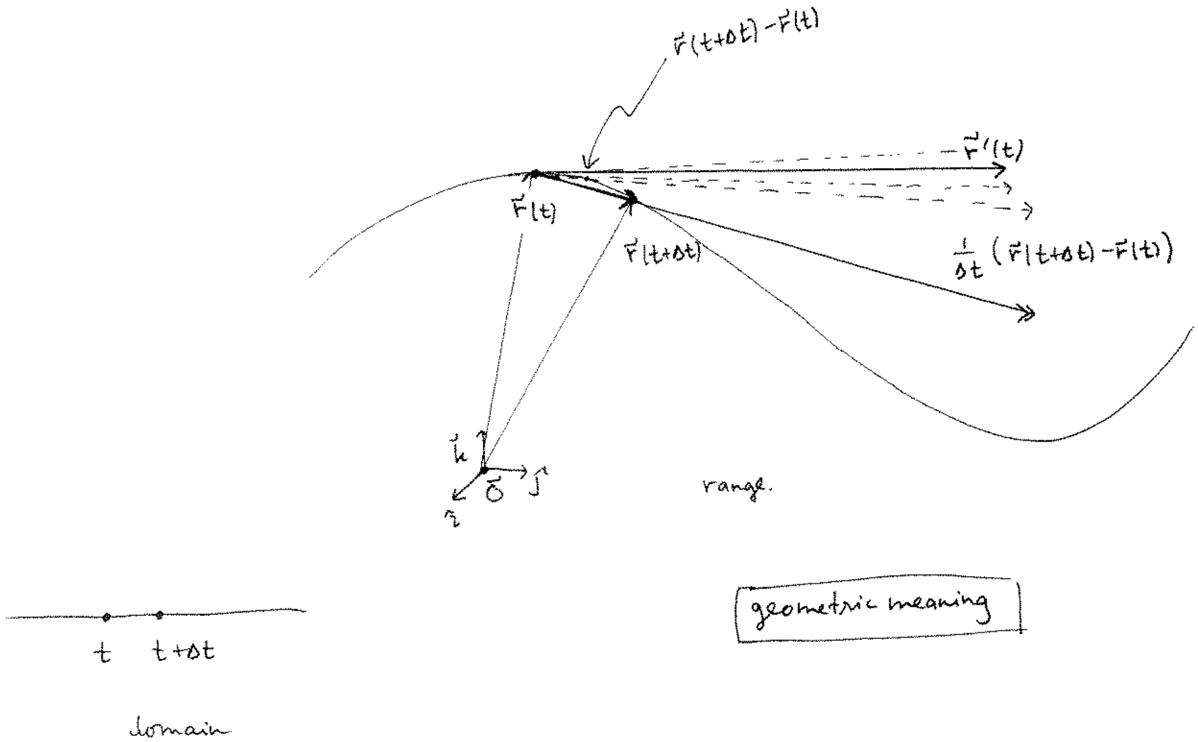
$$\frac{1}{\Delta t}\left[ f(t+\Delta t)\,\hat{i} + g(t+\Delta t)\,\hat{j} + h(t+\Delta t)\,\hat{k} - \left( f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} \right) \right]$$

$$= \lim_{\Delta t \to 0} \left(\frac{f(t+\Delta t) - f(t)}{\Delta t}\right)\hat{i} + \left(\frac{g(t+\Delta t) - g(t)}{\Delta t}\right)\hat{j} + \left(\frac{h(t+\Delta t) - h(t)}{\Delta t}\right)\hat{k}$$

(the limit of a vector expression is the vector which the expressions get arbitrarily close to as $\Delta t \to 0$, if it exists. this can be made precise with quantifiers "$\varepsilon$", "$\delta$", see p. 579)

$$= f'(t)\,\hat{i} + g'(t)\,\hat{j} + h'(t)\,\hat{k}$$

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle} \quad \boxed{\text{computation}}$$




$\boxed{\text{geometric meaning}}$

Example 6 go back to example 1. Compute $\dfrac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$ at $t = -1$, for $\Delta t = .1$, $\Delta t = -.1$

Then compute $\vec{r}\,'(t)$ ; $\vec{r}\,'(-1)$

Illustrate each computation on the page 1 graph paper