Math 2210 - 3

Fri 22 Jan

First finish Wed. notes on dot product (projection & work, pages 3-4.) Then...

Cross product §11.4

Unlike the dot product, $\vec{u}\cdot\vec{v}$, in which the value is a scalar (i.e. number), the value of the cross product, $\vec{u}\times\vec{v}$, is a vector, obtained by a process involving determinants

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Digress to determinants

2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$

these bars stand for determinant, not magnitude.

3×3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} := a_1 \underbrace{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}_{\text{det of "minor" matrix obtained by deleting row \& column in which } a_1 \text{ appears}} \overset{\text{note minus sign}}{-} a_2 \underbrace{\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}}_{\text{minor det by deleting row \& col of } a_2} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1 b_2 c_3 + a_2 c_1 b_3 + a_3 b_1 c_2 - a_1 c_2 b_3 - a_2 c_3 b_1 - a_3 c_1 b_2$$

Where do determinants come from?

- algebra related to matrices (det ≠ 0 if and only if matrix has an inverse)
- geometry of areas & volumes (as we shall see for 2×2 & 3×3).

example (1a) Find the area of the parallelogram spanned by

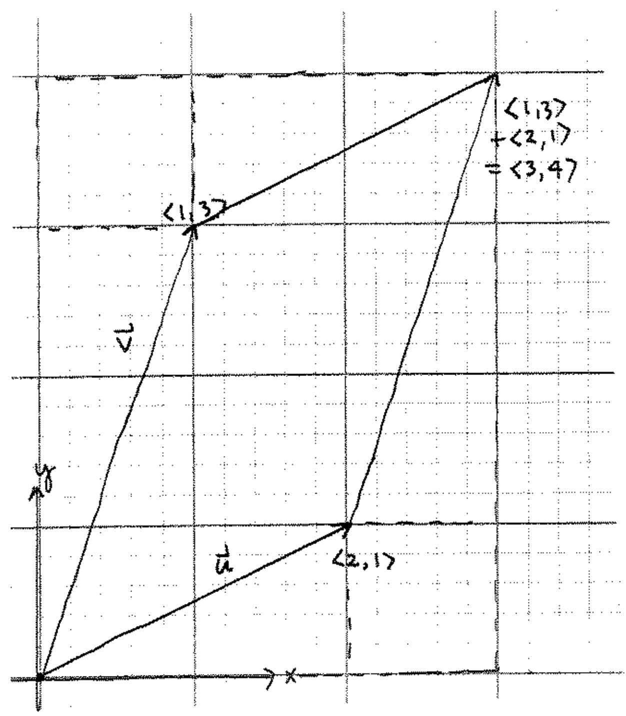
$\vec{u} = \langle 2,1\rangle$

$\vec{v} = \langle 1,3\rangle$.

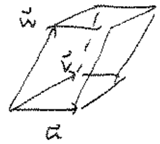
(1b) Compare to the two determinants

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} =$$



Fact: $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \pm$ area of  parallelogram $\left(\begin{array}{l} + \text{ if } \{\vec{u},\vec{v}\} \text{ are right-handed} \\ - \text{ if } \{\vec{u},\vec{v}\} \text{ are left-handed} \end{array}\right.$

$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \pm$ volume of  parallelepiped $\left(\begin{array}{l} + \text{ if } \{\vec{u},\vec{w},\vec{w}\} \text{ are right-handed} \\ - \text{ if } \{\vec{u},\vec{v},\vec{w}\} \text{ are left-handed.} \end{array}\right.$

(see page 4!)

~~end digress.~~

Cross product

$\vec{u} = \langle u_1, u_2, u_3 \rangle$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$$\vec{u}\times\vec{v} := \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat\imath - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat\jmath + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat k$$

$$= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$

$$= \langle u_2v_3 - v_2u_3,\ v_1u_3 - u_1v_3,\ u_1v_2 - v_1u_2 \rangle$$

example 2

Compute $\langle 2,1,0\rangle \times \langle 1,3,0\rangle$

Cross product algebra (some surprises!)

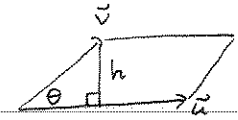
1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$. In particular, $\vec{u}\times\vec{u} = 0$!
2. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
   $(\vec{u}+\vec{w})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$
3. $c(\vec{u}\times\vec{v}) = (c\vec{u})\times\vec{v} = \vec{u}\times(c\vec{v})$
4. $\vec{u}\times\vec{0} = \vec{0}$
5. $(\vec{u}\times\vec{v})\cdot\vec{w} = \vec{u}\cdot(\vec{v}\times\vec{w})$; in fact $(\vec{u}\times\vec{v})\cdot\vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
6. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$
7. $\hat\imath\times\hat\jmath = \hat k$
   $\hat\jmath\times\hat k = \hat\imath$
   $\hat k\times\hat\imath = \hat\jmath$

Theorem (Gives geometric meaning of cross product)

Ⓐ $\vec{u}\times\vec{v}$ is $\perp$ to $\vec{u}$ and $\vec{v}$.

Also, $\{\vec{u},\vec{v},\vec{u}\times\vec{v}\}$ are right-handed

Ⓑ $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ = area of  spanned by $\vec{u},\vec{v}$



$h = |\vec{v}|\sin\theta$

so $A = |\vec{u}|h$
$= |\vec{u}||\vec{v}|\sin\theta$.

Ⓐ

check: $\vec{u}\cdot(\vec{u}\times\vec{v}) = u_1(u_2v_3 - v_2u_3) + u_2(v_1u_3 - u_1v_3) + u_3(u_1v_2 - v_1u_2) = 0$

$\vec{v}\cdot(\vec{u}\times\vec{v}) = 0$ analogous.

Ⓑ

check: depends on Lagrange identity, which is "fun" algebra (see HW)

$$|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2$$

Then,

$$= |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2\cos^2\theta$$
$$= |\vec{u}|^2|\vec{v}|^2\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}.$$

then take $\sqrt{\ }$'s! ∎

Example

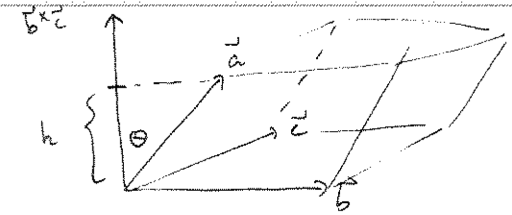
③ Check Theorem with example 2, $\langle 2,1,0\rangle \times \langle 1,3,0\rangle$.

Example ④ Recompute $\langle 2,1,0\rangle\times\langle 1,3,0\rangle = (2\hat{\imath}+\hat{\jmath})\times(\hat{\imath}+3\hat{\jmath})$ using algebra 1-7.

Theorem $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm$ volume of parallelepiped determined by $\vec{a}, \vec{b}, \vec{c}$

(+ vol if $\{\vec{a}, \vec{b}, \vec{c}\}$ right-handed, else − vol).

proof (right-handed case)



$$\begin{aligned} \text{Vol} &= (\text{area of base})(\text{height}) \\ &= |\vec{b}\times\vec{c}|\ |\vec{a}|\cos\theta \end{aligned}$$

(in case $\cos\theta > 0$, i.e. $\{\vec{a}, \vec{b}, \vec{c}\}$ right-handed. else opposite.)

but $\vec{a}\cdot(\vec{b}\times\vec{c}) = |\vec{a}|\,|\vec{b}\times\vec{c}|\cos\theta$

So $|\vec{a}|\cos\theta = \dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}$

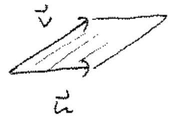
$$= \cancel{|\vec{b}\times\vec{c}|}\left(\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\cancel{|\vec{b}\times\vec{c}|}}\right)$$

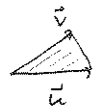
$$= \vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{a} \\ \hline \vec{b} \\ \hline \vec{c} \end{vmatrix} \quad \blacksquare$$

Example 5 Interpret $\begin{vmatrix} 0 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix}$ as a volume.

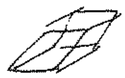
What we can do with cross product :

areas





volumes



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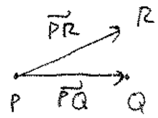
plane eqtns



$\langle a, b, c \rangle = \vec{PQ} \times \vec{PR}$

more!