Math 2210-3
Friday Jan. 15

- optional prob. session Tuesday SW 134, 10:45-11:45 ①
- further scaling back on hw due 1/20:
  in section 11.3 postpone all problems after #17. i.e., for 1/20 the only §11.3 problems are 1 ac e, 2b df 3b 4c 8 17

Recall geometric & algebraic ways of defining
- vectors
- vector addition
- scalar multiplication
- magnitude

to review this go through Wed notes page 2 carefully, i.e. 2abcd 3abcd.

i.e. less rushed than at end of Wed class! :)

Finish Wed notes:

page 3-4 vector algebra properties
unit vectors
the standard basis vectors $\hat{i}\ \hat{j}\ \hat{k}$

§11.3

Dot product (another algebraic operation on vectors with geometric meaning).

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

then $\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3$

"$\vec{a}$ dot $\vec{b}$"

$:=$ means "is defined to be"

(In $\mathbb{R}^n$, $\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$)

① $\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle =$

$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle =$

Dot product algebra:

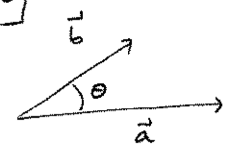
1. $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$ (so $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$)
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
   $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$
4. $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$
   (c a scalar!)
5. $\vec{a} \cdot \vec{0} = 0$

② Check some of these algebra rules
(to help you become comfortable using them)

The reason dot products are useful is that they let us do geometry. (As we will see.)
The geometric meaning of dot product is contained in the formula

<u>Theorem</u> : $\boxed{\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta}$

(where $\measuredangle \vec{a}, \vec{b} = \theta$:



Thus

(i) $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}$

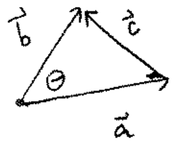
(ii) $\vec{a} \perp \vec{b}$ exactly if $\vec{a} \cdot \vec{b} = 0$ $\qquad$ $\perp$ means "is perpendicular to"

(3) Check that (i) gives the correct $\cos\theta$ (and θ) for $\vec{a} = \langle 6, 0 \rangle$, $\vec{b} = \langle 3, 3 \rangle$
(in practice you use this formula to find θ!)

(4) Show $P = (2,1,6)$
$Q = (4,7,9)$
$R = (8,5,-6)$
are vertices of a right triangle, by showing that 2 edges of ΔPQR are $\perp$!
(Use Theorem (ii))

Why the theorem on page 2 is true:

Consider the triangle



We will compute $|\vec{c}|^2$ two ways, set the expressions equal, and deduce $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ (after canceling terms)

(i) Law of cosines:

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

← generalizes Pythagorean Theorem, and is actually a consequence of it, see HW. (next assignment)

(ii) Vector computation:

$$\vec{c} = \vec{b} - \vec{a}$$

(because, $\vec{a} + (\vec{b}-\vec{a}) = \vec{b}$!)

so
$$
\begin{aligned}
|\vec{c}|^2 = \vec{c}\cdot\vec{c} &= (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a}) \\
&= \vec{b}\cdot(\vec{b}-\vec{a}) - \vec{a}\cdot(\vec{b}-\vec{a}) \quad (\text{dot prod algebra!}) \\
&= \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{a} \\
&= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}
\end{aligned}
$$

Thus (setting computation (i) = computation (ii))

$$\cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos\theta = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2\vec{a}\cdot\vec{b}$$

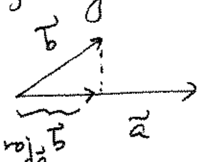
$$\cancel{-2}|\vec{a}||\vec{b}|\cos\theta = \cancel{-2}\,\vec{a}\cdot\vec{b}$$

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$ ∎

On Wednesday (next class) we'll use dot product to understand

1) the graph of $ax+by+cz=d$ is a plane, with $\perp$ vector $\langle a,b,c\rangle$

2) how to project one vector orthogonally onto another, using only algebra



3) Compute "work" when moving against constant force fields.