Name......Solutions......

I.D. number......

# Math 2210–3
# FINAL EXAM
May 4, 2010

This exam is closed–book. You are allowed a 4" by 6" index card of notes and formulas. You may also use a scientific calculator, but not one which is capable of doing integration or solving equations. Integral tables are included with this exam. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** This exam counts for 30% of your course grade. It has been written so that there are 150 points possible, and the point values for each problem are indicated in the right–hand margin. **Good Luck!**

1________________(30)

2________________(30)

3________________(20)

4________________(25)

5________________(15)

6________________(15)

7________________(15)

Total________________(150)

1a) Compute

$$\underbrace{\int_0^1 \int_{y-1}^{1-y}}\, 5 \, dx \, dy$$

(5 points)

$$\left. 5x \right]_{y-1}^{1-y} = 5\left(1-y-(y-1)\right)$$
$$= 5(2-2y) = 10(1-y)$$
$$= \int_0^1 10(1-y)\,dy = 10\left(y-\frac{y^2}{2}\right)\Big]_0^1$$
$$= 10\left(\frac{1}{2}\right) = 5$$

1b) Sketch the region of integration for the iterated integral in (1a). (Hint: it's a triangle.) Label the boundary segments with the equations they satisfy, and label the three vertices with their coordinates.

(5 points)

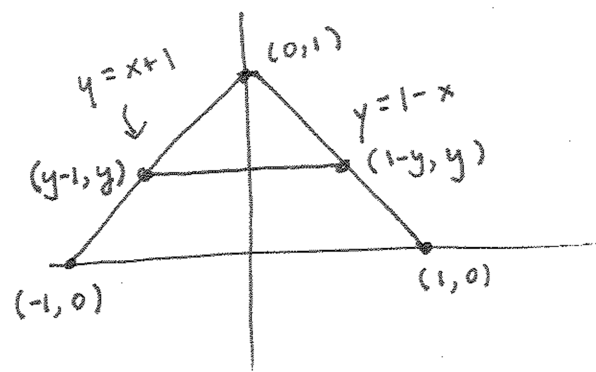
$0 \le y \le 1$

for each $y$

$y-1 \le x \le 1-y$

$\uparrow$ $= \text{if } y = x+1$

$\uparrow$ $= \text{if } y = 1-x$

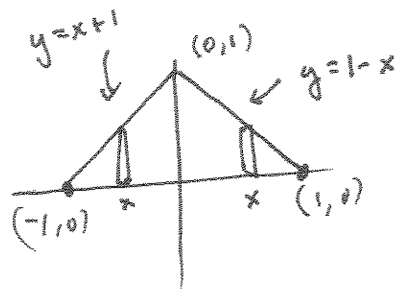



1c) Use geometry and your triangle from (1b) to verify that your answer to (1a) is correct.

(5 points)

$$\iint 5\, dA = 5 \cdot \text{area} = 5 \cdot \frac{1}{2} bh = \frac{5}{2} \cdot 2 \cdot 1 = 5 \checkmark$$

1d) Compute the double integral in (1a) over the same triangular region, but with the order of integration reversed. Hint: You will need to break the integral into two pieces. (8 points)




$$\iint_\Delta 5\,dA = \int_{-1}^{0}\int_0^{x+1} 5\,dy\,dx + \int_0^1\int_0^{1-x} 5\,dy\,dx$$

$$= \int_{-1}^{0} 5(x+1)\,dx + \int_0^1 5(1-x)\,dx$$

$$= 5\left(\frac{x^2}{2}+x\right)\Big]_{-1}^{0} + 5\left(x-\frac{x^2}{2}\right)\Big]_0^1$$

$$= 5\left(0-\left(\tfrac{1}{2}-1\right)\right) + 5\left(1-\tfrac{1}{2}\right)$$

$$= \frac{5}{2}+\frac{5}{2} = 5 \quad ✓$$

1e) Find the centroid of this triangular region, assuming constant density . You may use symmetry to deduce one of the center of mass coordinates. (7 points)

if we let $\delta = 1$ then mass = area = 1

$\bar{x} = 0$ by symmetry.

$$\bar{y} = \frac{\iint 1\,y\,dA}{\text{area}} \leftarrow \int_0^1 \underbrace{\int_{y-1}^{1-y} y\,dx}\,dy$$

$$xy\Big]_{x=y-1}^{1-y} = y\left(1-y-(y-1)\right)$$

$$= y(2-2y)$$

$$= 2y(1-y)$$

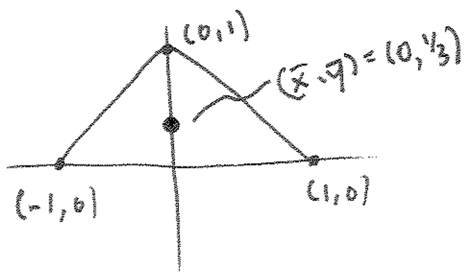
$$= 2(y-y^2)$$

$$= 2\int_0^1 y-y^2\,dy$$

$$= 2\left(\frac{y^2}{2}-\frac{y^3}{3}\right)\Big]_0^1 = 2\left(\tfrac{1}{2}-\tfrac{1}{3}\right) = \frac{2}{6} = \frac{1}{3}$$

So $\bar{y} = \dfrac{1/3}{1} = \dfrac{1}{3}$

$$\boxed{\text{centroid } (0, 1/3)}$$



2) Consider the parametric curve with position vector given by

$$\mathbf{r}(t) = \langle 3 - t\,,\, e^{2t} \rangle.$$

There is a sketch of part of this curve below.

2a) This parametric curve has range which lies on the graph $y = f(x)$ of some function $f(x)$. Find a formula for $f(x)$.

(4 points)

$x = 3 - t$

$t = 3 - x$

$y = e^{2t} = \boxed{e^{2(3-x)} = e^{6} e^{-2x} = f(x)}$

2b) Find $\mathbf{r}(0)$, $\mathbf{r}'(0)$, and $\mathbf{r}''(0)$. Find the unit tangent and normal vectors, $\mathbf{T}$ and $\mathbf{N}$, when $t = 0$. Carefully indicate the point with position $\mathbf{r}(0)$ in the picture below, and add the remaining four vectors to the picture, making sure to draw them as accurately as possible and in the appropriate locations, using a piece of paper or index card ruler – length and direction both count!

(14 points)

$\vec{r}(t) = \langle 3-t, e^{2t} \rangle$

$\vec{r}\,'(t) = \langle -1, 2e^{2t} \rangle$

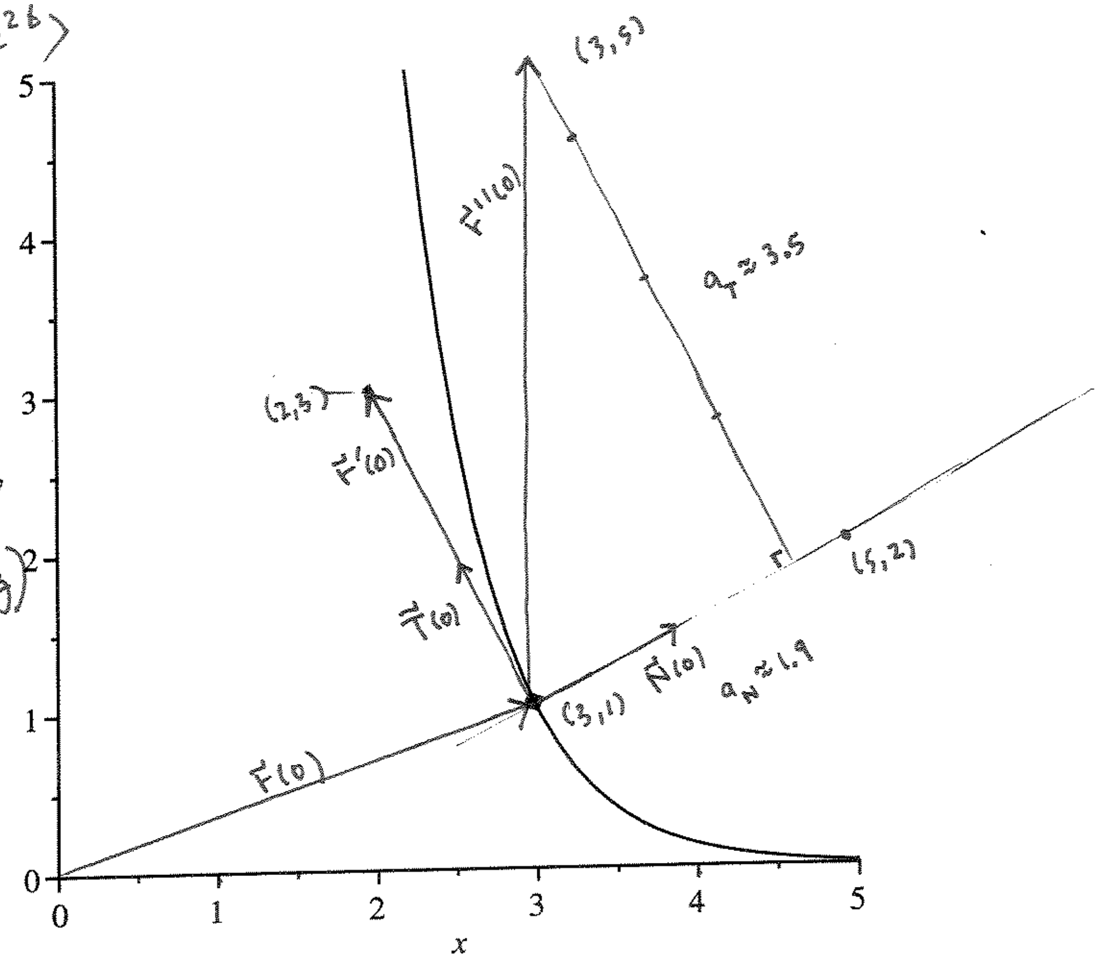
$\vec{r}\,''(t) = \langle 0, 4e^{2t} \rangle$

$\boxed{\begin{array}{l} \vec{r}(0) = \langle 3, 1 \rangle \\ \vec{r}\,'(0) = \langle -1, 2 \rangle \\ \vec{T}(0) = \frac{1}{\sqrt{5}} \langle -1, 2 \rangle \end{array}}$

so $\boxed{\vec{N}(0) = \frac{1}{\sqrt{5}} \langle 2, 1 \rangle}$

($\perp$ $\vec{T}$ & in dir. of bending)

$\boxed{\vec{r}\,''(0) = \langle 0, 4 \rangle}$



2c) Use the graph on the previous page, an appropriate right triangle which you add to the picture, and an index card ruler to measure the components of the acceleration $\mathbf{r}''(0)$ in the tangential and normal directions. Label which component is which. Also record the components below (accuracy within 0.3 suffices).

(6 points)

$a_T \approx 3.5$

$a_N \approx 1.9$

2d) Find the exact values of the acceleration components in the tangential (T) and normal (N) directions, when t = 0. You can either use the "Roller coaster equation", or the dot product to compute these components! (Your values should be close to the numerical approximations in (2c).)

(6 points)

$a_T = \vec{r}''(0) \cdot \vec{T} = \langle 0, 4\rangle \cdot \frac{1}{\sqrt{5}}\langle -1, 2\rangle = \boxed{\frac{8}{\sqrt{5}}}$

$\approx 8 \cdot \frac{4}{9} = \frac{32}{9}$

$\approx 3.6$

$$\begin{array}{r} 2.3 \\ \underline{2.3} \\ 69 \\ \underline{46\phantom{0}} \\ 5.29 \end{array} \qquad \begin{array}{r} 2.2 \\ \underline{2.2} \\ 44 \\ \underline{44\phantom{0}} \\ 4.84 \end{array}$$

$\sqrt{5} \approx 2.25 = \frac{9}{4}$

$a_N = \vec{r}''(0) \cdot \vec{N} = \langle 0, 4\rangle \cdot \frac{1}{\sqrt{5}}\langle 2, 1\rangle$

$\boxed{= \frac{4}{\sqrt{5}}} \approx 1.8$

$$\begin{array}{r} 3.6 \\ 9\overline{)32} \\ \underline{27} \\ 5.0 \end{array}$$

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much harder to use REE:

$\vec{r}''(t) = \frac{dv}{dt}\vec{T} + \kappa v^2 \vec{N}$

$v = \|\vec{r}'(t)\| = \sqrt{1 + 4e^{4t}}$ @ $t=0$ $v = \sqrt{5}$

$\frac{dv}{dt} = \frac{1}{2}(1+4e^{4t})^{-1/2} 16e^{4t}$

@ $t=0$ $\frac{dv}{dt} = \frac{1}{2}\frac{1}{\sqrt{5}} \cdot 16 = \boxed{\frac{8}{\sqrt{5}}}$ ✓

$\kappa = \frac{|y''x' - x''y'|}{(x'^2 + y'^2)^{3/2}}$

@ $t = 0$. $\kappa = \frac{|4 \cdot (-1) - 0|}{5^{3/2}} = \frac{4}{5^{3/2}}$

So $\kappa v^2 = 5 \cdot \frac{4}{5^{3/2}} = \boxed{\frac{4}{\sqrt{5}}}$ ✓

3) Consider the function

$$T(x, y) = 70 + (x - 2y)\cos(x^2 - 9).$$

3a) Compute the gradient of $T(x, y)$, and its value at the point $(x, y) = (3, 1)$.

(5 points)

$$\nabla T = \langle T_x, T_y \rangle = \langle 1 \cdot \cos(x^2-9) + (x-2y)(-\sin(x^2-9)) \cdot 2x,$$
$$-2\cos(x^2-9) + 0 \rangle$$

$$\boxed{\nabla T = \langle \cos(x^2-9) + 2x(x-2y)(-\sin(x^2-9)),\ -2\cos(x^2-9) \rangle}$$

$$\nabla T(3,1) = \langle \cos 0 + 6(1)(-0),\ -2\cos 0 \rangle$$

$$\boxed{\nabla T(3,1) = \langle 1, -2 \rangle}$$

3b) Suppose that $T(x, y)$ represents the temperature (Farenheit degrees) at location $x$ miles east and $y$ miles north of a fixed reference point. Suppose a coyote is following the path with the position vector from problem 2, $\mathbf{r}(t) = \langle 3 - t, e^{2t} \rangle$ at time t hours. Notice at time $t = 0$ the coyote is at location (3, 1). How fast is the temperature the coyote is experiencing on his travels changing at $t = 0$? Include correct units.

(10 points)

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = 1(-1) + (-2)(2) \quad \text{deg/mile} \cdot \text{mile/h}$$

$$\boxed{= -5 \ \text{deg/hour}}$$

@ $t=0$ $x = 3$ mile $\quad \frac{dx}{dt} = -1$ mile/h

$y = 1$ mile $\quad \frac{dy}{dt} = 2$ mile/h

$\frac{\partial T}{\partial x} = 1$ deg/mile

$\frac{\partial T}{\partial y} = -2$ deg/mile

(or $\nabla T(\vec{r}(0)) \cdot \vec{r}\,'(0) = \langle 1, -2 \rangle \cdot \langle -1, 2 \rangle = -5$ deg/hour.)

3c) Use differentials to approximate $T(2.98, 1.01)$, using $T(3, 1) = 71$. (Hint: the exact value is close to 70.95314.)

(5 points)

$$T(x+dx, y+dy) \approx T(x,y) + T_x\,dx + T_y\,dy$$

$(x,y) = (3,1)$ $\qquad = 71 + 1(-.02) - 2(.01)$

$T(3,1) = 70 + (3-2)\cos 0$

$= 71$ ✓ $\qquad = 71 - .04$

$$\boxed{= 70.96}$$

$T_x(3,1) = 1 \quad dx = -.02$

$T_y(3,1) = -2 \quad dy = .01$

4a) Show that the vector field

$$F(x, y) = \langle \overset{M}{e^x \cos(y) + 2x}, \overset{N}{-e^x \sin(y) + 3y} \rangle$$

is a gradient field by checking an appropriate partial derivative condition.

(5 points)

$N_x = M_y$ ?

$N_x = -e^x \sin y$

$M_y = e^x(-\sin y)$

$-e^x \sin y = e^x(-\sin y)$ ✓

since (scalar) curl is zero, this is a gradient field.

4b) Find a function $f(x, y)$ whose gradient is the vector field $F(x, y)$ above.

(5 points)

$f_x = e^x \cos y + 2x$

$\Rightarrow f = \int f_x \, dx = e^x \cos y + x^2 + C(y)$

$f_y = \cancel{e^x(-\sin y)} + C'(y) \underset{\uparrow}{=} \cancel{-e^x \sin y} + 3y$

from $\vec{F}$ must = N

$C'(y) = 3y$

$C(y) = \frac{3}{2}y^2 + C$

$$\boxed{f(x,y) = e^x \cos y + x^2 + \frac{3}{2}y^2} \quad (+c)$$

4c) Use the chain rule and the definition of line integral to show that if $\boldsymbol{G}$ is any gradient field with $\nabla g = \boldsymbol{G}$ then for any curve C connecting point A to point B,

$$\int_C \boldsymbol{G}(\boldsymbol{r}) \cdot d\boldsymbol{r} = g(B) - g(A)\,.$$

(5 points)

Let $\mathcal{C}$ be parameterized by $\vec{r}(t)$, $a \le t \le b$.

Then $\displaystyle\int_{\mathcal{C}} \vec{G}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$$= \int_a^b \underbrace{\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{= \frac{d}{dt} g(\vec{r}(t)) \text{ by chain rule}}\,dt$$

$$= g(\vec{r}(t))\Big]_{t=a}^{b} \qquad \text{FTC (1 variable)}$$

$$= g(\vec{r}(b)) - g(\vec{r}(a)) = g(B) - g(A).$$

4d) Let C be the horizontal line segment of the x–axis connecting (0,0) to (1,0). For F and f as in parts 4ab above, compute the line integral

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

and verify that it does in fact equal $f(1,0) - f(0,0)$.

(10 points)

$$\int_C M\,dx + N\,dy = \int_{x=0}^{1} e^x \cos(0) + 2x\,dx + (\underbrace{-e^x \sin 0 + 3\cdot 0}_{0})\,\underbrace{dy}_{0}$$

curve is $\vec{r}(x) = \langle x, 0\rangle$

so $dy = 0$.

$$= \int_0^1 e^x + 2x\,dx = e^x + x^2\Big]_0^1$$

$$\boxed{= e+1-1 = e}$$

$$f(x,y) = e^x \cos y + x^2 + \tfrac{3}{2}y^2$$

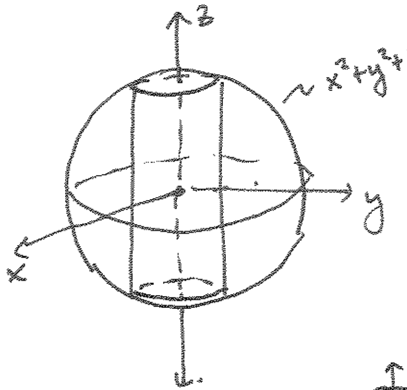
$f(1,0) = e+1$

$f(0,0) = 1$

$f(1,0) - f(0,0) = e$ ✓

5a) Set up an iterated triple integral in rectangular coordinates which will yield the volume of the region which is bounded inside the vertical cylinder of radius 1 about the z–axis, and the radius 3 sphere centered at the origin. Do not evaluate this integral.

(5 points)

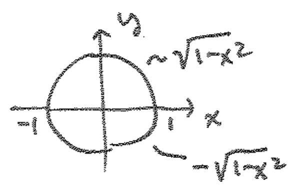



$-1 \le x \le 1$

for $x$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

for $x$ & $y$, $-\sqrt{9-x^2-y^2} \le z \le \sqrt{9-x^2-y^2}$

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} 1 \, dz\,dy\,dx.$$




5b) Compute the volume of the region in (5a) using an appropriate choice of spherical, cylindrical, or polar coordinates.

(10 points)

spherical: $0 \le r \le 1$

$0 \le \theta \le 2\pi$

for $r, \theta$ $\quad -\sqrt{9-r^2} \le z \le \sqrt{9-r^2}$

$dV = r\,dr\,d\theta\,dz$

$$V = \int_0^1\int_0^{2\pi}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} 1 \, \underbrace{r\,dz\,d\theta\,dr}_{dV}$$

$$= \int_0^1\int_0^{2\pi} 2r\sqrt{9-r^2}\, r\,d\theta\,dr$$

(what you'd get with polar)

$$= \int_0^1 2\pi \cdot 2r\sqrt{9-r^2}\,dr$$

$$= 4\pi \int_0^1 r\sqrt{9-r^2}\,dr$$

$u = 9-r^2$

$du = -2r\,dr$

$-\frac{du}{2} = r\,dr$

$\int u^{1/2}\left(-\frac{du}{2}\right)$

$= -\frac{1}{2}\frac{2}{3}u^{3/2}$

$= -\frac{1}{3}u^{3/2}$

$$= 4\pi\left(-\tfrac{1}{3}(9-r^2)^{3/2}\right)\Big]_0^1$$

$$= \frac{4\pi}{3}\left[-8^{3/2} + 9^{3/2}\right]$$

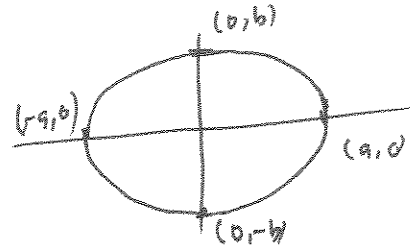
$8^{3/2} = (2^3)^{3/2} = 2^{9/2} = 2^{4\frac{1}{2}}$

$= 16\sqrt{2}$

$$\boxed{= \frac{4}{3}\pi\left(27 - 16\sqrt{2}\right)}$$

6) Let $a$ and $b$ be positive numbers. Consider the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$




Consider the vector field $F = \langle M, N \rangle = \langle -y, x \rangle$.

6a) Compute the line integral

$$\int_C F(r) \cdot dr$$

where C is the ellipse, traversed once counter–clockwise. (Hint: parameterize the ellipse analogously to how you would parameterize a circle.)

(10 points)

$\vec{r}(t) = \langle a\cos t, b\sin t \rangle$

$0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -a\sin t, b\cos t \rangle$

(pts lie on ellipse since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1$ )

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \underset{\uparrow \atop -y}{-b\sin t}, \underset{\uparrow \atop x}{a\cos t} \rangle \cdot \langle -a\sin t, b\cos t \rangle \, dt$$

$$= \int_0^{2\pi} \underbrace{ab\sin^2 t + ab\cos^2 t}_{ab(\sin^2 t + \cos^2 t) = ab} \, dt = \int_0^{2\pi} ab \, dt = ab(2\pi) = \boxed{2\pi ab}$$

6b) Use Green's Theorem and your work in 6a to deduce that the area enclosed by the ellipse is given by the formula

$$A = \pi a b.$$

(5 points)

Green: $\displaystyle\oint_{\substack{C \\ \partial(\text{ellipse})}} M\,dx + N\,dy = \iint_{\text{ellipse}} (N_x - M_y)\, dA = \iint_{\text{ellipse}} \underbrace{1 - -1}_{2} \, dA = 2(\text{area})$

$N = x$

$N_x = 1$

$M = -y$

$M_y = -1$

So $\underset{\uparrow \atop (6a)}{2A = 2\pi ab}$, $\div 2 \Rightarrow \boxed{A = \pi ab}$

7a) Consider the inverse square force field

$$F(r) = \frac{r}{\|r\|^3}$$

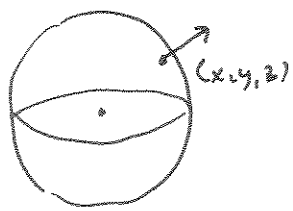
Consider the sphere S of radius $a > 0$, centered at the origin. Show that the flux integral

$$\iint_S F \cdot n \, dS$$

always equals $4\pi$, independently of radius $a$. Hint: both **F** and the unit normal **n** are scalar multiples of the position vector on these spheres. (5 points)

on sphere of radius a

unit exterior normal $= \frac{1}{a}\langle x,y,z\rangle = \vec{n}$ at $(x,y,z)$.




$\vec{F} = \frac{1}{a^3}\langle x,y,z\rangle$

So $\vec{F}\cdot\vec{n} = \frac{1}{a}\langle x,y,z\rangle \cdot \frac{1}{a^3}\langle x,y,z\rangle = \frac{1}{a^4}(x^2+y^2+z^2)$

$= \frac{a^2}{a^4} = \frac{1}{a^2}$

So $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \frac{1}{a^2}\,dS$

$= \frac{1}{a^2}$ (area of sphere)

$= \frac{1}{a^2}(4\pi a^2) = 4\pi$ ✓

7b) Writing **F** in terms of (x,y,z) we have

$$F(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle.$$

Compute the divergence of F, and verify that it equals zero. (5 points)

$M = x(x^2+y^2+z^2)^{-3/2}$

$M_x = 1(x^2+y^2+z^2)^{-3/2} + x(-\tfrac{3}{2})(x^2+y^2+z^2)^{-5/2}\cdot 2x$

$= (x^2+y^2+z^2)^{-3/2}\left(1 - \frac{3x^2}{x^2+y^2+z^2}\right)$

by symmetry

$N_y = (x^2+y^2+z^2)^{-3/2}\left(1 - \frac{3y^2}{x^2+y^2+z^2}\right)$

$P_z = (x^2+y^2+z^2)^{-3/2}\left(1 - \frac{3z^2}{x^2+y^2+z^2}\right)$

So $\operatorname{div}\vec{F} = M_x + N_y + P_z = (x^2+y^2+z^2)^{-3/2}\left(\underbrace{1+1+1 - 3\left(\frac{x^2+y^2+z^2}{x^2+y^2+z^2}\right)}_{0}\right)$

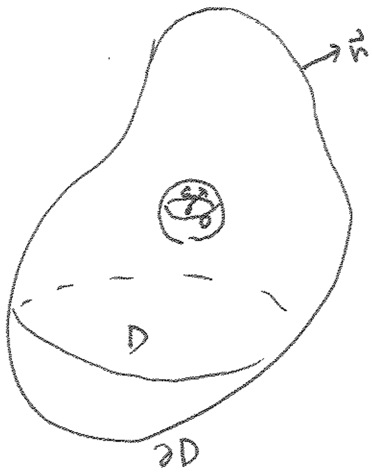
$= 0$

7c) Now let D be any three−dimensional domain containing the origin and for which the divergence theorem applies. Apply this theorem to the domain obtained from D by deleting a small ball of radius $a$, centered about the origin and use your result from 7a, to deduce that the flux integral

$$\iint_{\partial D} \boldsymbol{F} \cdot \boldsymbol{n} \, dS$$

also has value $4\pi$, independent of the domain D.

(5 points)




on the deleted domain

$$\iint_{\partial(\text{domain with ball deleted})} \vec{F}\cdot\vec{n}\,dS = \iiint_{\text{deleted domain}} \operatorname{div}\vec{F}\,dV = 0 \text{ by (7b)}$$

$$\| $$

$$\iint_{\partial D} \vec{F}\cdot\vec{n}\,dS + \underbrace{\iint_{\partial(\text{Ball})} \vec{F}\cdot\vec{n}\,dS}_{= -4\pi}$$

this normal is the inner unit normal to the deleted ball so get opposite ans to 7a

so $\displaystyle\iint_{\partial D} \vec{F}\cdot\vec{n}\,dS - 4\pi = 0$

i.e. $\displaystyle\iint_{\partial D} \vec{F}\cdot\vec{n}\,dS = 4\pi$ ∎