Math 2210-3
Friday Feb. 5

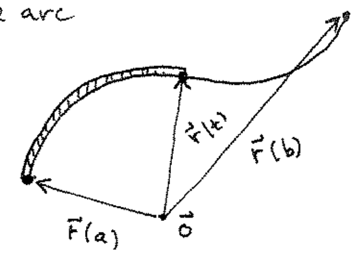
§ 11.7: Geometry of curves and physics of particle motion

Curvature of curves

Let a parametric curve have position vector $\vec{r}(t)$, $a \le t \le b$, with $\|\vec{r}\,'(t)\| > 0 \ \forall t$ (positive speed)

- Consider the arclength function $s(t)$:

$$s(t) = \int_a^t \|\vec{r}\,'(\tau)\| \, d\tau \quad = \text{length of the curve arc corresponding to } a \le \tau \le t$$



Since $\frac{ds}{dt} = \|\vec{r}\,'(t)\| > 0$ (FTC)

$s(t)$ is a strictly increasing function and has an inverse function $t(s)$.

Also

$$\frac{dt}{ds} = \frac{1}{\left(\frac{ds}{dt}\right)} = \frac{1}{v}$$

(where we use the letter v for the speed $\|\vec{r}\,'\|$.)
$\|\vec{v}\|$

- Consider the unit tangent vector $\vec{T} := \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{v}\vec{r}\,'(t)$
- The curvature ("κ") of a curve measures the rate of change of $\vec{T}$ with respect to arclength:

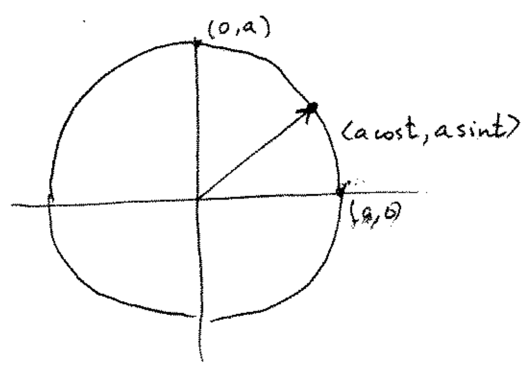
$$\boxed{\kappa := \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right\| = \frac{1}{v} \left\| \frac{d\vec{T}}{dt} \right\|}$$

notice you can parameterize a curve however you want, you'll always get the same κ at the same pts of the curve!

Exercise 1 : A circle of radius a has curvature $\frac{1}{a}$ (everywhere)

Check this!

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$ (is easiest parameterization)
$= a\langle \cos t, \sin t\rangle$



$\vec{r}\,'(t) =$

$v =$

$\vec{T} =$

$\vec{T}\,'(t) =$

$\kappa =$

What do you think the curvature of a straight line is?

<u>Exercise 2</u> The curvature of any helix $\vec{r}(t) = \langle a\cos t, a\sin t, bt \rangle$ is also constant. What is its value?

<u>Exercise 3</u> Consider the curve with position vector

$$\vec{r}(t) = e^t \langle \cos t, \sin t \rangle.$$

(it's an exponentially growing spiral, as t increases)

3 a) Find $\vec{r}\,'(t)$

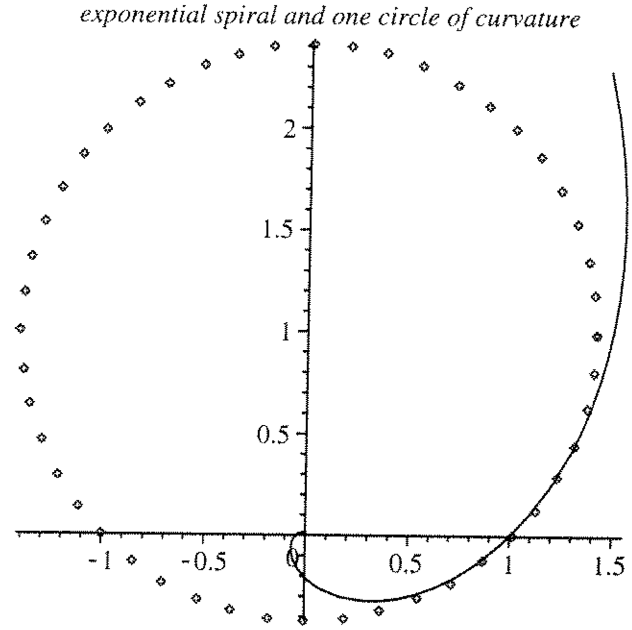
$\vec{T}(t)$

$\vec{N}(t)$

b) Compute the curvature of the curve at $\vec{r}(0)$.

c) Find the center & radius of the "circle of curvature" at this point – i.e. the circle passing through this point, tangent to the curve there, and with same curvature. Hint: your algebra should agree with the picture!

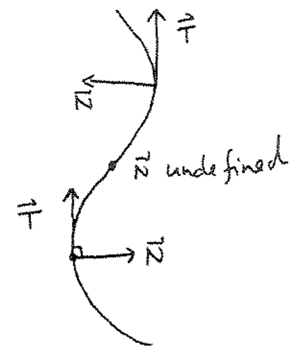
exponential spiral and one circle of curvature



We use curvature in:

RCE

Physics of particle motion ("roller coaster equation" for $\vec{r}''(t)$)



looking at $\vec{T}(t+\Delta t)-\vec{T}(t)$ deduce $\vec{N}$ points in direction curve is bending

- unit normal vector $\vec{N}$:

$$\vec{T}(t)\cdot\vec{T}(t)\equiv 1$$

$$\frac{d}{dt}:\qquad 2\,\vec{T}'(t)\cdot\vec{T}(t)\equiv 0$$

$$\vec{N} := \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{\kappa v}\,\vec{T}'(t)$$ unit normal to curve (defined when $\kappa\neq 0$).

- RCE:

$$\frac{\vec{r}'(t)}{v} = \vec{T}$$

$$\vec{r}'(t) = v\vec{T}$$

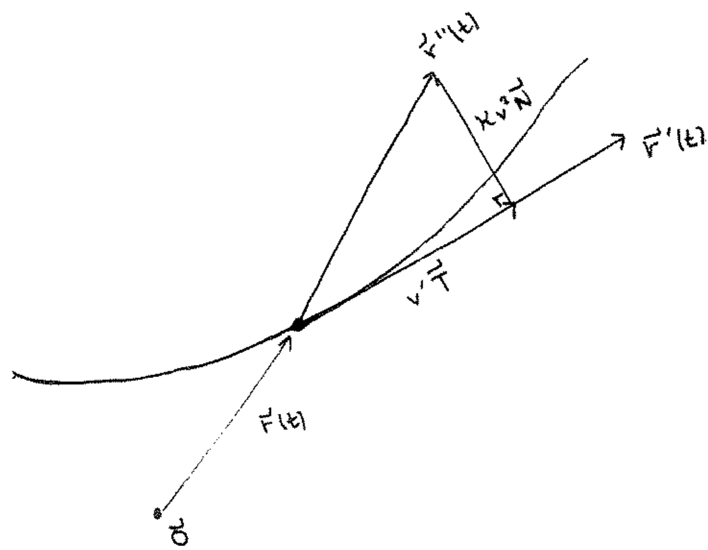
$$\vec{r}''(t) = v'\vec{T} + v\,\underbrace{\vec{T}'(t)}_{\kappa v \vec{N}}$$

$$\boxed{\vec{r}''(t) = v'\vec{T} + \kappa v^2\vec{N}}$$ ← Roller Coaster Equation

tangential component of acceleration indicates whether you're speeding up or slowing down

normal component of acceleration is proportional to κ, and to the square of the speed. Note $\kappa = \frac{1}{R}$, where R is radius of circle of curvature.

(note formula in book writes $\frac{ds}{dt}$ for speed v and $\frac{d^2s}{dt^2}$ for v'.)

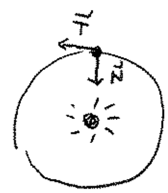


<u>Exercise 4</u> For a particle moving along a circle of radius R, the RCE reads

$$\vec{r}''(t) = v'(t)\,\vec{T} + \frac{v^2}{R}\vec{N}$$

Show that for circular, constant speed orbits of planets about the sun, Kepler's 3rd law ($T^2 = kR^3$), k independent of the planets, is equivalent to the inverse square law

$$\vec{r}''(t) = -\frac{c}{R^2}\vec{N} \quad , \; c \text{ independent of the planets.}$$



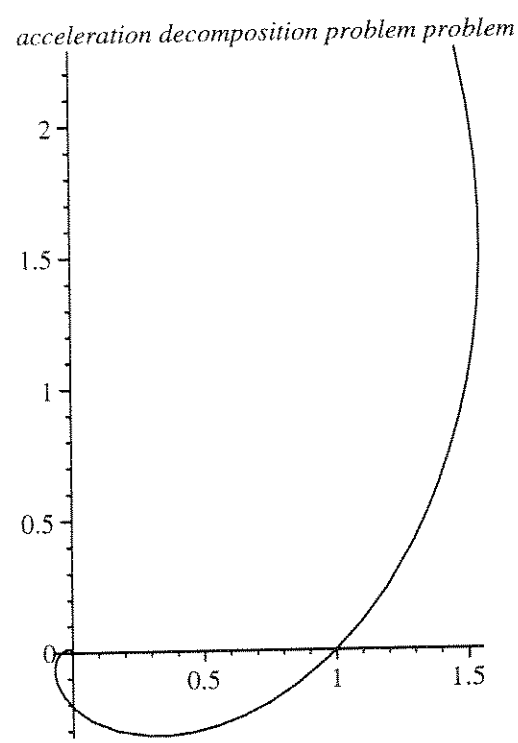
<u>Exercise 5</u> Consider $\vec{r}(t) = e^t\langle\cos t, \sin t\rangle$ from exercise 3

a) compute $\vec{r}''(0)$ (you already computed $\vec{T}(0), \vec{N}(0)$)

b) Use a folded piece of paper marked with the same units as below, to draw $\vec{r}''(0)$ as a vector starting at $\vec{r}(0)$. Also include $\vec{T}(0), \vec{N}(0)$. Use your ruler to measure the components of $\vec{r}''(0)$ in the $\vec{T}$ & $\vec{N}$ directions

c) Compute the exact $\vec{T}$ & $\vec{N}$ components of $\vec{r}''(0)$ with dot product – compare to 2b)

d) Use RCE for $\vec{T}$ & $\vec{N}$ components of $\vec{r}''(0)$ – should agree with 2c)



Math 2210–3
Homework Set 4: due February 10

(underlined bold italics problems are to be handed in)

11.7: ***1, 4, 9, 12, 17, 18, 28, 36***, 48, 55, ***58, 59, 61, 86.***
11.8: ***2, 5, 7, 8, 9, 13, 14, 29.***

Extension of problem 42 section 11.7:
We consider the parametric curve

$$\mathbf{r}(t) = < t^2, t >.$$

42a) Show this curve lies on the parabola with equation $x = y^2$.

42b) Compute the point with position vector $\mathbf{r}(1)$. Then Compute $\mathbf{r}'(1)$, $\mathbf{r}''(1)$. Plot the point and these vectors appropriately and carefully onto the picture of the range curve below.
42c) Using a ruler which you construct using the scale below, draw a picture which decomposes $\mathbf{r}''(1)$ into its tangential and normal pieces. Measure the length of each piece to determine numerical values of the tangential and normal components of acceleration.
42d) Find the unit tangent and normal vectors $\mathbf{T}(1), \mathbf{N}(1)$, analytically. Then use the dot product to compute the components of acceleration in these two directions. Compare with the numerical values in (42c).
42e) Use the RCE equation to recompute the components in (42d), by computing $\kappa(1)$, $v(1)$, $v'(1)$. You should get the same answers!