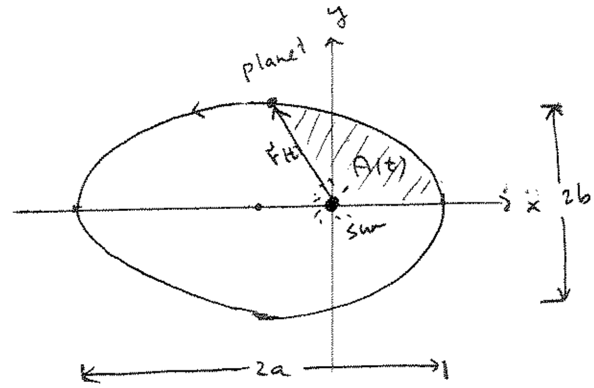
Math 2210-3
Wednesday Feb. 3
§ 11.4-11.5

HW for 2/10
TBA

Kepler's Laws and Newton's amazing deduction

with data from Tycho Brahe, Johannes Kepler formulated the following empirical laws of planetary motion



I. Planets orbit the sun in ellipses, with the sun at a focus of the ellipse

II. A line segment from the focus sun to the planet (i.e. $\vec{r}(t)$ if sun is origin) sweeps out area at a constant rate ("equal areas in equal times")

III. The square of an orbit's period is proportional to the cube of the ellipse's major axis (same proportionality constant for all planetary orbits)

$$T^2 = k a^3$$

Kepler's Laws

Newton's Laws

Newton was able to explain Kepler's Laws with

$$\vec{F} = m\vec{r}''(t)$$

and

$$\vec{F} = \frac{GmM}{r^2}\left(-\frac{\vec{r}}{r}\right)$$

m = planet mass
M = sun mass
G = universal const
$r = \|\vec{r}\|$.

unit vector from planet to sun
(sun feels opposite force, but since $M \gg m$ it hardly accelerates)

which reduces to

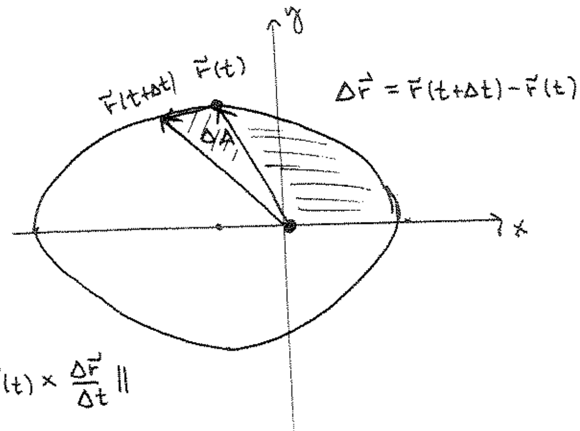
$$\boxed{\vec{r}''(t) = -\frac{c}{r^2}\left(\frac{\vec{r}}{r}\right)}$$

The text explains how NL $\Rightarrow$ KL.
But the profound science is that Kepler's empirical laws imply that planetary acceleration must be towards the sun, and that the strength of the acceleration satisfies the inverse square law

In other words, Newton deduced the inverse square law of gravitational attraction as a mathematical consequence of Kepler's Laws. ~ he never actually directly measured planetary to sun gravitational attraction !!

Newton's deductions

Step 1: II Equal areas in equal time with planar orbit)
holds if and only if $\vec{r}''(t)$ is parallel to $\vec{r}$, i.e.
planet accelerates towards sun:



in time Δt,
the growth in area ΔA is

$$\Delta A \approx \underbrace{\tfrac{1}{2}\,\|\vec{r}(t) \times \Delta\vec{r}\,\|}_{\text{triangle area}}$$

$$\frac{\Delta A}{\Delta t} \approx \frac{1}{2\,\Delta t}\,\|\vec{r}(t) \times \Delta\vec{r}\,\| = \frac{1}{2}\,\Big\|\vec{r}(t) \times \frac{\Delta\vec{r}}{\Delta t}\Big\|$$

$\lim_{\Delta t \to 0}$: $\quad \text{const} \overset{\text{II}}{=} \dfrac{dA}{dt} = \dfrac{1}{2}\,\|\vec{r} \times \vec{r}\,'\|$

but $\vec{r} \times \vec{r}\,'$ is a positive multiple of $\hat{k}$, so in fact

$$\boxed{\vec{r} \times \vec{r}\,' = \vec{C}} \quad (= \text{const}\,\hat{k})$$

take $\dfrac{d}{dt}$: $\quad \underbrace{\vec{r}\,' \times \vec{r}\,'}_{0} + \vec{r} \times \vec{r}\,'' = \vec{0} \quad \Rightarrow \quad \boxed{\vec{r}\,'' \parallel \vec{r}}$

(all steps reversible)

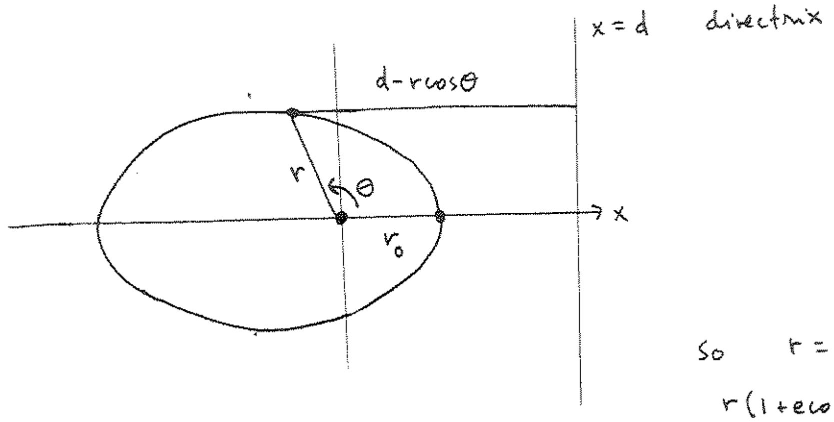
1b)
For later, compute $\dfrac{dA}{dt} = C_1$ in terms of orbital data; i.e. the period T & the ellipse:

$$\int_0^T \underbrace{\frac{dA}{dt}}_{C_1}\,dt = \text{ellipse area} = \pi ab$$

$$C_1 T = \pi ab$$

$$\boxed{\frac{dA}{dt} = C_1 = \frac{\pi ab}{T}} \qquad \boxed{\|\vec{r} \times \vec{r}\,'\| = 2C_1 = \frac{2\pi ab}{T}}$$

Step 2 polar coord. form of an ellipse (or other conic) with focus at origin



$$\frac{r}{d - r\cos\theta} = e \quad \text{(eccentricity const.)}$$

So $\quad r = e(d - r\cos\theta)$

$$r(1 + e\cos\theta) = ed$$

$$\boxed{r = \frac{ed}{1 + e\cos\theta}}$$

$e < 1$ ellipse
$e > 1$ hyperbola

Step 3 rectangular (x-y) eqtn for same ellipse ~ details shown for your convenience.

$$\frac{\sqrt{x^2+y^2}}{d-x} = e$$

$$\sqrt{x^2+y^2} = e(d-x)$$

$$x^2+y^2 = e^2(d^2 - 2dx + x^2)$$

$$x^2(1-e^2) + 2e^2dx + y^2 = e^2d^2$$

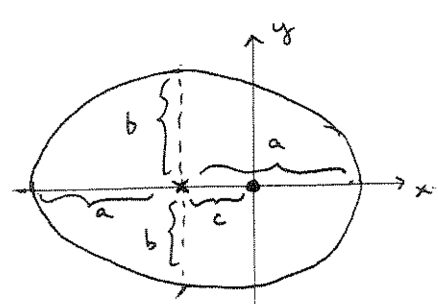
$$(1-e^2)\left(x^2 + \frac{2e^2}{1-e^2}dx\right) + y^2 = e^2d^2$$

$$(1-e^2)\left(x + \frac{de^2}{1-e^2}\right)^2 + y^2 = e^2d^2 + \frac{d^2e^4}{1-e^2}$$

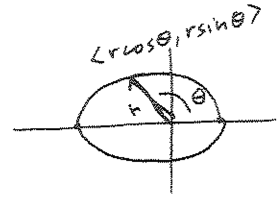
$$= \frac{e^2d^2}{1-e^2}$$

$$\boxed{\frac{\left(x + \frac{de^2}{1-e^2}\right)^2}{\left(\frac{e^2d^2}{(1-e^2)^2}\right)} + \frac{y^2}{\left(\frac{e^2d^2}{1-e^2}\right)} = 1}$$

$$\boxed{a = \frac{ed}{1-e^2}, \quad b = \frac{ed}{\sqrt{1-e^2}}, \quad c = \frac{de^2}{1-e^2}}$$



Step 4 : $\vec{r}\,'$, $\vec{r}\,''$ computations using polar coordinates



$\vec{r}(t) = r\langle \cos\theta, \sin\theta, 0\rangle$ in $\mathbb{R}^3$

$r\langle \cos\theta, \sin\theta\rangle$ in $\mathbb{R}^2$ $\quad r = r(t)$, $\theta = \theta(t)$)

so $\vec{r}\,'(t) = r'\langle \cos\theta, \sin\theta, 0\rangle + r\theta'\langle -\sin\theta, \cos\theta, 0\rangle$

check!

thus $\vec{r}\times\vec{r}\,' = \vec{r}\times\left(\underbrace{\cancel{\tfrac{r'}{r}\vec{r}}}_{0} + r\theta'\langle -\sin\theta, \cos\theta, 0\rangle\right) = r^2\theta'\,\hat{k}$

deduce from Step (1b)

$$\boxed{r^2\theta' = 2C_1 = \frac{2\pi ab}{T}}$$

continue to $\vec{r}\,''$:

$$\vec{r}\,''(t) = r''\langle \cos\theta, \sin\theta, 0\rangle + r'\theta'\langle -\sin\theta, \cos\theta, 0\rangle$$
$$+ r'\theta'\langle -\sin\theta, \cos\theta, 0\rangle + r\theta''\langle -\sin\theta, \cos\theta, 0\rangle$$
$$+ r(\theta')^2\langle -\cos\theta, -\sin\theta, 0\rangle$$

$$\vec{r}\,''(t) = \left(r'' - r(\theta')^2\right)\underbrace{\langle \cos\theta, \sin\theta\rangle}_{\text{radial unit vector}} + \left(2r'\theta' + r\theta''\right)\underbrace{\langle -\sin\theta, \cos\theta\rangle}_{\perp \text{ to radial}}$$

since acceleration is radial (step 1), this is its strength, and this is zero.

So, write $f = r'' - r(\theta')^2$. We (Newton) hope $f = \frac{c}{r^2}$, i.e. $r^2 f$ is constant?

Compute!

$$r^2 f = r^2 r'' - r^3(\theta')^2$$
$$= r^2 r'' - \frac{1}{r}\left(r^2\theta'\right)^2$$
$$r^2 f = r^2 r'' - \frac{1}{r}\left(2C_1\right)^2 \qquad \text{from box above}$$

almost done!

$$r^2 f = r^2 r'' - \frac{1}{r}(2C_1)^2$$ from page 4 want this to be const !!

$$r = \frac{ed}{1+e\cos\theta}$$ step 2

$$r(1+e\cos\theta) = ed$$

$$r'(1+e\cos\theta) - re\theta'\sin\theta = 0$$

$$r'\left(\frac{ed}{r}\right) - re\theta'\sin\theta = 0$$

$$r'\cancel{e}d - \underbrace{r^2\theta'}_{2C_1}\cancel{e}\sin\theta = 0$$

$$r' = \frac{2C_1}{d}\sin\theta$$

$$r'' = \frac{2C_1}{d}\theta'\cos\theta$$

So

$$r^2 f = r^2\left(\frac{2C_1}{d}\theta'\right)\cos\theta - \frac{1}{r}(2C_1)^2$$

$$= (2C_1)^2\left[\frac{\cos\theta}{d} - \frac{1}{r}\right]$$

$$= (2C_1)^2\left[\frac{\cos\theta}{d} - \frac{1+e\cos\theta}{ed}\right]$$

$$= (2C_1)^2\left(-\frac{1}{ed}\right)$$ a constant!

Step3!!

$$= 4\pi^2\frac{a^2\cancel{b^2}}{T^2}\left(-\frac{a}{\cancel{b^2}}\right)$$

$$r^2 f = -\frac{a^3}{T^2}4\pi^2$$

$$= -k4\pi^2$$ k from Kepler's 3rd law III !

$$f = -\frac{4\pi^2 k}{r^2}$$ independent of planet!

$$\boxed{\vec{r}'' = -\frac{4\pi^2 k}{r^2}\frac{\vec{r}}{\|\vec{r}\|}}$$

Kepler's observations $\Longrightarrow$ Newton's theory of gravitation!