Math 2210-3
Friday Feb. 26

§12.4 Differentiability

What should it mean for a function

$f(x,y)$ to be differentiable at (x_0,y_0)?
$f(x,y,z)$ " " (x_0,y_0,z_0)?

Need a definition which is useful for Calculus to follow.
Return to our roots; 1-variable calculus

1. $f(x)$ is differentiable at x_0 means

$$* \quad \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} := f'(x_0) \text{ exists}$$

→ this definition won't work if $\vec{x}_0, \vec{h}$ are vectors (can't divide by a vector!)

→ reformulate def'n to get rid of denom:

$*$ is equivalent to

$$\frac{f(x_0+h)-f(x_0)}{h} = m + \varepsilon(h)$$

where $m = f'(x_0)$
and $\varepsilon(h) \to 0$ as $h \to 0$

i.e.

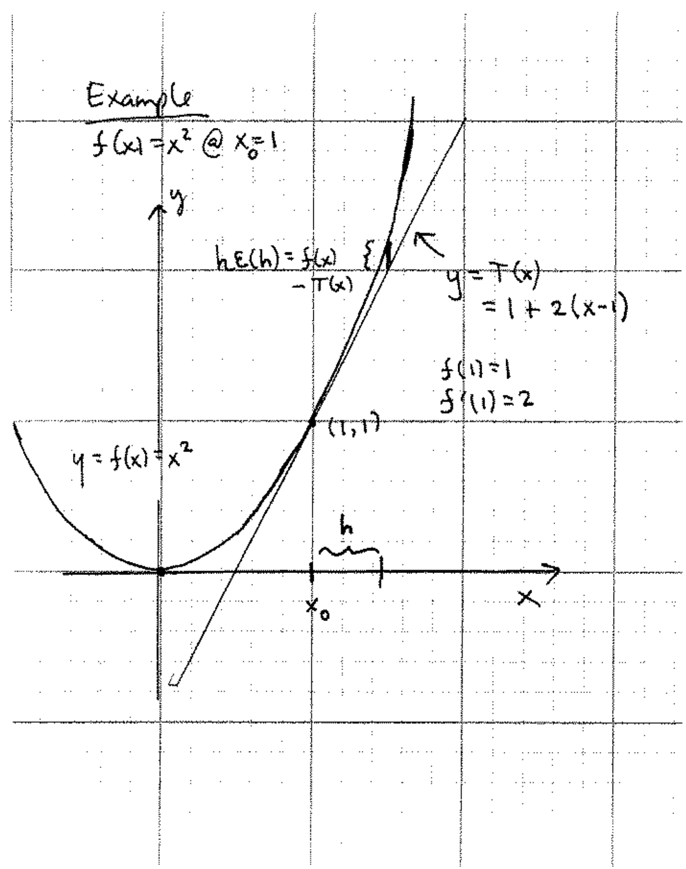
$$f(x_0+h) - f(x_0) = mh + h\,\varepsilon(h)$$

denominator gone! → $\boxed{f(x_0+h) = f(x_0) + mh + h\,\varepsilon(h), \quad \varepsilon(h)\to 0 \text{ as } h\to 0}$

write $x = x_0+h$:
$x - x_0 = h$

$$\boxed{f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{\text{Tangent approx } T(x) \text{ to } f(x)} + \underbrace{h\,\varepsilon(h)}_{\text{Error between } f(x)\ \&\ T(x)}}$$

Error between $f(x)$ & $T(x)$, → 0 faster than h, since $\varepsilon(h)\to 0$ too, as $h \to 0$.



Thus

2 $f(x,y)$ is differentiable at (x_0,y_0) means

f has a good linear approximation (i.e. graph is a plane) there:

$$\boxed{f(x_0+h_1, y_0+h_2) = f(x_0,y_0) + f_x(x_0,y_0)h_1 + f_y(x_0,y_0)h_2 + h_1\varepsilon_1(\vec{h}) + h_2\varepsilon_2(\vec{h}) \quad \text{where } \varepsilon_1, \varepsilon_2 \to 0 \text{ as } \vec{h}\to 0}$$

Looks better in vector notation

$\vec{p}_0 := \langle x_0, y_0\rangle$

$\vec{h} := \langle h_1, h_2\rangle$

$\nabla f := \langle f_x, f_y\rangle$

$\vec{\varepsilon}(\vec{h}) := \langle \varepsilon_1(h), \varepsilon_2(h)\rangle$

$$\boxed{f(\vec{p}_0+\vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(\vec{h}) \quad \vec{\varepsilon}(\vec{h})\to 0 \text{ as } \vec{h}\to 0}$$

- just like page 1
- works in any dimension

rewrite, if you wish

$\langle x_0,y_0\rangle = \vec{p}_0$

$\langle x,y\rangle = \vec{p}_0+\vec{h}$

$$f(x,y) = \underbrace{f(x_0,y_0) + \nabla f(x_0,y_0)\cdot\langle x-x_0, y-y_0\rangle}_{:= T(x,y)} + \underbrace{\vec{h}\cdot\vec{\varepsilon}(\vec{h})}_{\text{error}}$$

the tangent approx to f at (x_0,y_0)

(its graph $z=T(x,y)$ is tangent plane to graph $z=f(x,y)$, at $(x_0,y_0,f(x_0,y_0))$.

<u>Example</u> Find the tangent approximation to $f(x,y) = 4-x^2-y^2$ at $(x_0,y_0)=(0,1)$

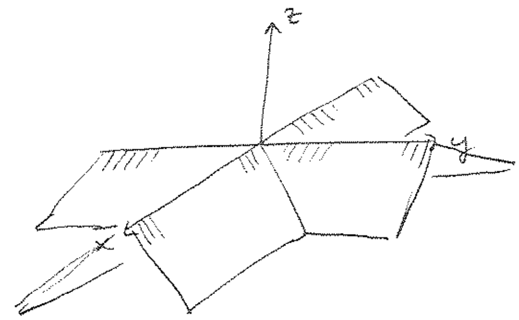
Sketch the graph $z=f(x,y)$ and its tangent plane $z=T(x,y)$

<u>ans</u> $z = T(x,y) = 3 + 0(x-0) - 2(y-1)$

$z = 5-2y$

Example Consider the "rooftop" function

$$f(x,y) = -\left(\text{minimum of } |x|, |y|\right)$$
$$= -\left(\text{distance from } (x,y) \text{ to the } x \text{ or } y \text{ axis}\right)$$



graph $z = f(x,y)$

Do $f_x(0,0)$, $f_y(0,0)$ exist?
Is $f(x,y)$ differentiable at $(0,0)$?

Luckily, life is usually good:

Theorem If f_x and f_y are continuous functions near (x_0,y_0), then f is differentiable at (x_0,y_0)

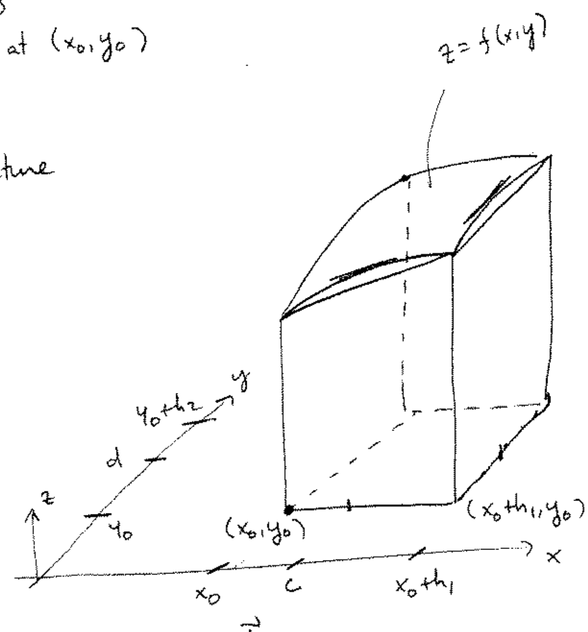
Explanation: you can understand this using the mean value theorem, from 1210. Consult picture at right:

$$f(x_0+h_1, y_0+h_2) - f(x_0+h_1, y_0) = \left(\frac{\partial f}{\partial y}(x_0+h_1, d)\right) h_2$$
$$f(x_0+h_1, y_0) \quad - f(x_0,y_0) = \left(\frac{\partial f}{\partial x}(c, y_0)\right) h_1$$

add

$$f(\vec{p}_0+\vec{h}) - f(\vec{p}_0) = \langle f_x(c,y_0), f_y(x_0+h_1,d)\rangle \cdot \vec{h}$$

$$f(\vec{p}_0+\vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h}$$
$$+ \langle \underbrace{f_x(c,y_0) - f_x(x_0,y_0)}_{\varepsilon_1}, \underbrace{f_y(x_0+h_1,d) - f_y(x_0,y_0)}_{\varepsilon_2} \rangle \cdot \vec{h}$$



MVT! need f_x, f_y exist nearby.

if f_x, f_y are continuous near $\vec{p}_0$ then $\varepsilon_1, \varepsilon_2 \to 0$ as $\vec{h} \to \vec{0}$

<u>Example</u> Where <u>is</u> the rooftop function differentiable?

<u>Example</u> : Was $f(x,y) = 4 - x^2 - y^2$ differentiable at $(0,1)$?
Why is this $f(x,y)$ differentiable everywhere?

<u>Example</u> : Show $f(x,y,z)$ is differentiable everywhere, $f(x,y,z) = e^x \sin 2y + 3z + 1$
let $(x_0, y_0, z_0) = (0,0,0)$.
Find the linear (tangent) approximation to f at this point.

Compare $f(x,y,z)$ to $T(x,y,z)$

at $x = .1$, $y = -.1$, $z = .1$

at $x = .01$, $y = -.01$, $z = .01$

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> f:=(x,y,z)->exp(x)*sin(2*y)+3*z+1;
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$$f := (x, y, z) \to e^x \sin(2\,y) + 3\,z + 1$$

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> f(.1,-.1,.1);
  f(.01,-.01,.01);
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1.080436433
1.009800343

<u>Theorem</u> : If f is differentiable at $\vec{p}_0$
<u>then</u> f is continuous at $\vec{p}_0$

(Converse not true: see roof function)

<u>Reason</u> : $f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0) = \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{\varepsilon}(\vec{h}) \cdot \vec{h}$

so $|f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0)| \le \|\nabla f(\vec{p}_0)\| \|\vec{h}\| + \|\vec{\varepsilon}(\vec{h})\| \|\vec{h}\| \to 0$ as $\vec{h} \to 0$

so $\lim_{\vec{p} \to \vec{p}_0} f(\vec{p}) = f(\vec{p}_0)$ ∎

used
- triangle inequality $\|\vec{a} + \vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$
- Cauchy-Schwarz inequality $|\vec{a} \cdot \vec{b}| \le \|\vec{a}\| \|\vec{b}\|$