Hw for Wed Apr 14
14.1 1, 2, 10, 11, 15, 17, 19, 20abc, 21, 25
14.2 1, 4, 7, 8, 15, 17, 18, 21, 25, 26

Math 2210-3
Friday April 9
Chptr 14: Vector Calculus & multivariable FTC ← pinnacle of multivariable Calc + key topics in science & engineering, for PDE's & ODE's.

§ 14.1 Vector fields ← e.g. force fields + more

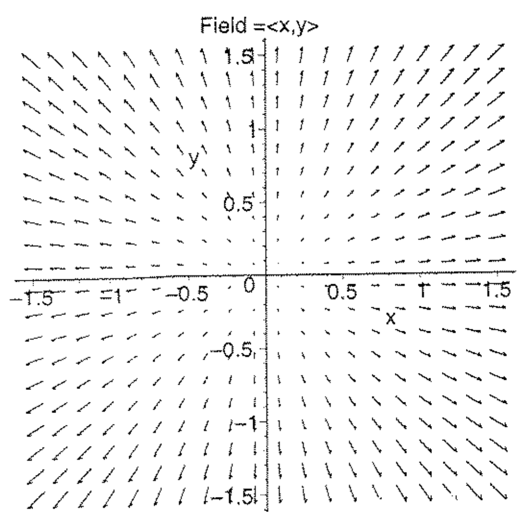
a vector field assigns a vector to each point in a domain (with the same dimensions, usually)

$n=2$: $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$  $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$
$n=3$: $\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$

- We "graph" a vector field by drawing sample (output) vectors at a collection of domain points.
- Special vector fields are actually the gradients of functions, $\vec{F} = \nabla f$. In this case we know the vector fields will be $\perp$ to level sets (level curves or surfaces) and will point in direction of greatest rate of increase for $f$.

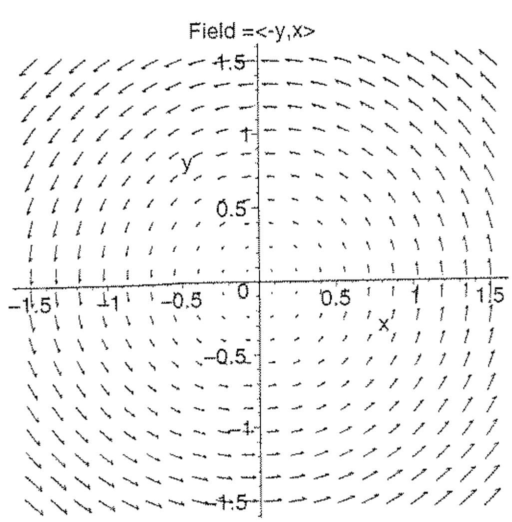
Examples! (talk about drawing these by hand too.)

1



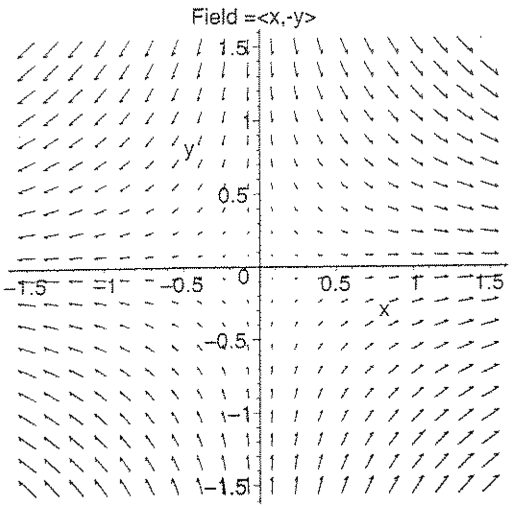

$\vec{F} = \langle M(x,y), N(x,y)\rangle = \langle x, y\rangle$
gradient field? ($\vec{F} = \nabla f$)
What are the level curves of $f$?

2




$\langle M, N\rangle = \langle -y, x\rangle$
gradient field?
(Think Escher!)

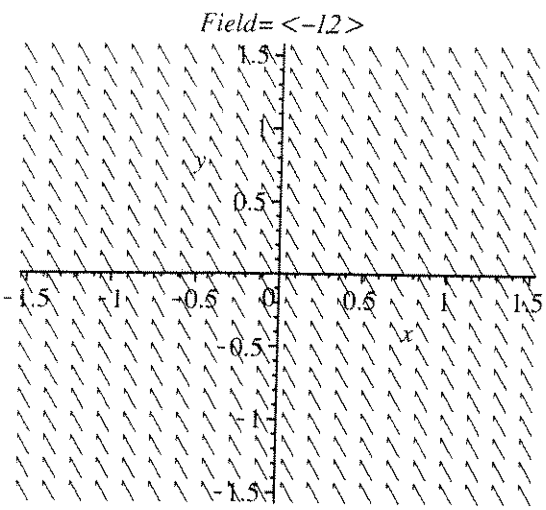
3




$\langle M, N \rangle = \langle x, -y \rangle$

gradient field?

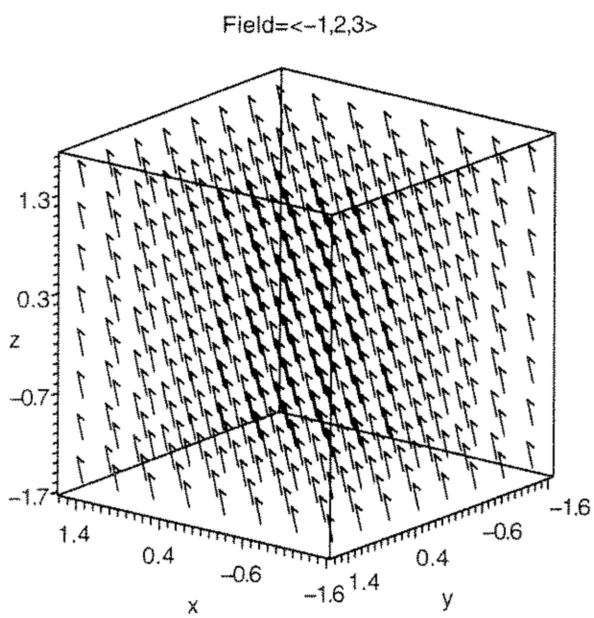
4




$\langle M, N \rangle = \langle -1, 2 \rangle$

gradient field?

5



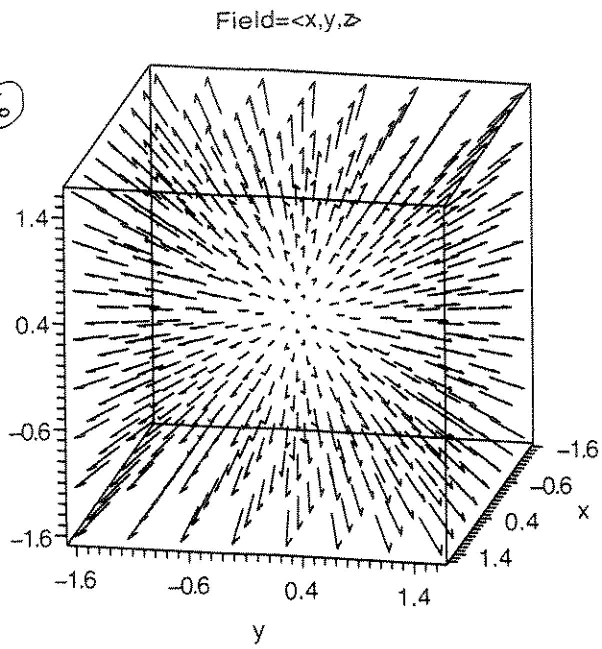

$\langle M, N, P \rangle = \langle 1, 2, 3 \rangle$

gradient field?

f?

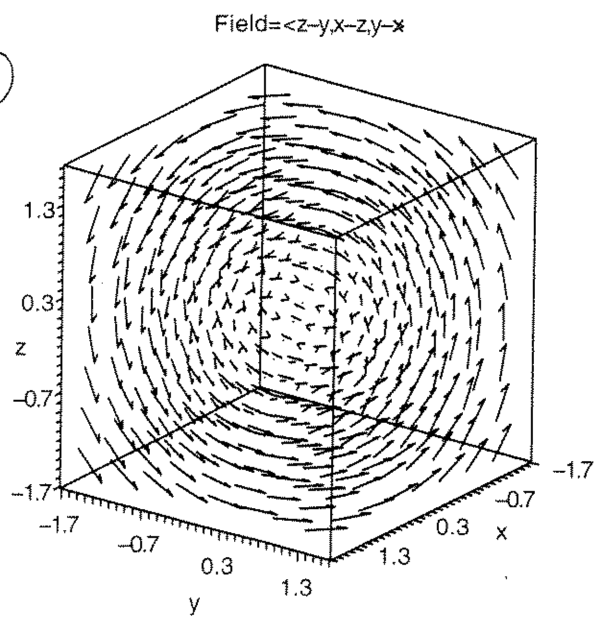
level surfaces of f?

6




$\langle M, N, P\rangle = \langle x, y, z\rangle$
gradient field?

7




$\langle M, N, P\rangle = \langle z-y, x-z, y-x\rangle$
gradient field?

8 Exercise

If $\langle a,b,c\rangle$ is a vector,
compute $\vec{F}(x,y,z) = \langle x,y,z\rangle \times \langle a,b,c\rangle$
and sketch the vector field!
gradient field?

Some important derivative operations of vector fields:

Think of $\nabla$ meaning $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle = \langle \partial_x, \partial_y, \partial_z \rangle$ in $\mathbb{R}^3$

$\boxed{\text{divergence}}$ or $\langle \partial_x, \partial_y \rangle$ in $\mathbb{R}^2$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle = M_x + N_y + P_z \quad \text{in } \mathbb{R}^3$$

$$= \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \rangle \cdot \langle M, N \rangle = M_x + N_y \quad \text{in } \mathbb{R}^3$$

$\boxed{\text{curl}}$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \quad \text{in } \mathbb{R}^3$$

in $\mathbb{R}^2$, scalar $\underset{\operatorname{curl} \vec{F}}{\text{curl}}$ is $N_x - M_y$

9

Exercise Compute $\nabla \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$ for a sample of our example fields, thinking about the English meaning of "divergence" and "curl"
Include 8

Theorem If $\vec{F}$ is a gradient field, $\vec{F} = \nabla f$, then $\operatorname{curl} \vec{F} = \vec{0}$. (The converse is true locally, as we'll see later)

proof: If $\vec{F} = \langle f_x, f_y, f_z \rangle$ (or $\langle f_x, f_y \rangle$ for $n=2$) then

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$$

$= \langle 0, 0, 0 \rangle$ by equality of cross-partials! ∎

"Theorem" (later sections make this precise)

- $\operatorname{div} \vec{F}$ measures whether vector fields illustrate expansion or contraction at a point (or near a point), if $\vec{F}$ is interpreted as a velocity vector field for a field of moving particles (precisely, whether the "volume" of a domain of particles increases or decreases)
- $\operatorname{curl} \vec{F}$ measures "circulation" of such a velocity vector field

10 Exercise (if we have time).

The most important ~~se~~ function in physics is

$$f(x,y,z) = -\frac{1}{\sqrt{x^2+y^2+z^2}} \qquad \left(= -\frac{1}{\varrho}\right)$$

which is (up to scaling) the potential energy function for a mass (i.e. gravity) or a charge (i.e. electricity) at the origin.

- Compute $-\nabla f = \vec{F}$

  notice this is the force field (inverse square law) for gravitational & electric attraction.

- Thus $\nabla \times \vec{F} = \vec{0}$
- Check that $\nabla \cdot \vec{F} = 0$ too (except at $\vec{x} = \vec{0}$)

  (this is one of your HW exercises!)