Math 2210-3
Review Sheets
April 28, 2010

Tuesday May 4
10:30 - 12:30
(you'll have until 1:00).
in our classroom.

Problem session
Monday May 3
10 am - noon,
LCB 215

- Final exam is comprehensive, 30% of total grade. ←
- In order to receive a grade of C or higher, you must earn a grade of C or higher on the final exam. If you meet ~~th~~is criterion your final course grade is obtained by adding your weighted scores (20% for each midterm, 30% each for HW & final), and using the summed curve points for each component.
- Approximate percentages on exam, by chapter:

  11: 20-25% vectors and curves
  12: 20-25% derivative concepts for functions of several variables
  13: 25-30% multiple integration and applications
  14: 25-30% vector calculus

- You may bring one 4" by 6" index card to the exam, as well as a scientific (i.e. NOT GRAPHING) calculator. I'll provide integral tables.

Topics Overview (See also, old midterm review sheets.)

Basic tools: each tool is defined algebraically, has algebraic properties, as well as geometric properties and significance

vectors: addition, scalar multiplication,
magnitude
unit vectors

dot product: def., algebra
geometric meaning, $\perp$ vectors, eqtns of planes
projections and components

cross product: def, algebra
geometric meanings
applications to area & other geometry.

lines, planes, conics, quadric surfaces, cylinders

polar, cylindrical, spherical coordinates

level curves in $\mathbb{R}^2$
Level surfaces in $\mathbb{R}^3$

Differentiation

parametric curves

$\vec{r}(t)$, $\vec{r}'(t)$, $\vec{r}''(t)$: computation and meaning

going from a level curve description (in the plane) to a parametric curve, & vice verse

differentiation rules (sums, products, chain)

$\vec{T}$, $\vec{N}$, κ

$\vec{r}''(t) = v'(t)\vec{T} + \kappa v^2 \vec{N}$; computing and/or measuring ($v = \|r'\|$ = speed, κ = curvature)

$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{r}'(t)\Delta t$ tangent approximation

functions of several variables

$f(\vec{x}+\Delta\vec{x}) \approx f(\vec{x}) + \underbrace{\nabla f(\vec{x})\cdot\Delta\vec{x}}_{\text{"}df\text{"}}$ tangent approximation (= differential approx.)

$D_{\vec{u}} f(\vec{x}) = \lim_{t\to 0} \frac{f(x+t\vec{u}) - f(\vec{x})}{t} = \nabla f(\vec{x})\cdot\vec{u} = \|\nabla f(x)\|\cos\theta$; calculate & estimate

Chain rule (combine tangent functions for $f(\vec{x})$ and $\vec{r}(t)$ to deduce)

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot\vec{r}'(t) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt} \quad \text{etc.}$$

$\nabla f(\vec{x}) \perp$ to level curve of f thru $\vec{x}$ (or level surface) (eqtns for tangent planes)

Max-min problems: by finding critical points (perhaps after constraint elimination) or via Lagrange multipliers

parametric surfaces,

$\vec{r}(u,v)$

$\vec{r}(u+\Delta u, v+\Delta v) \approx \vec{r}(u,v) + \vec{r}_u(u,v)\Delta u + \vec{r}_v(u,v)\Delta v$ tangent approximation

unit normal $\vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$ $\left(= \frac{1}{\sqrt{1+f_x^2+f_y^2}}\langle -f_x, -f_y, 1\rangle \text{ if } \vec{r}(x,y) = \langle x, y, f(x,y)\rangle \right)$

vector fields

$\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$

$\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$

$$\text{div}\,\vec{F} = \underset{\substack{\uparrow \\ \langle\partial_x,\partial_y\rangle \text{ or} \\ \langle\partial_x,\partial_y,\partial_z\rangle}}{\vec{\nabla}}\cdot\vec{F} = \begin{matrix} M_x + N_y & \mathbb{R}^2 \\ M_x + N_y + P_z & \end{matrix}$$

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$$

$\mathbb{R}^2$, scalar curl of $\langle M, N\rangle = N_x - M_y$.

Integration

Basic integration

$\int_a^b f(x)\,dx$

double or triple integrals over
rectangles or coordinate boxes

more complicated iterated integrals in $\mathbb{R}^2$, $\mathbb{R}^3$

domain $\leftrightarrow$ setting up iterated integrals

area, volume, mass, center of mass, moments of inertia
(memorize or write down formulas!)

Polar coords
$dA = r\,dr\,d\theta$

Cylindrical coords
$dV = r\,dr\,d\theta\,dz$

Spherical coords
$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

More complicated integrals

ALL based on small scale (tangent) approximation and parameterization

$$\int_C f(\vec{x})\,ds := \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

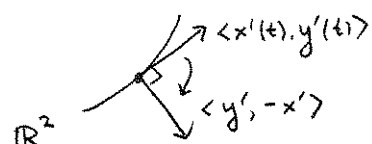
also written as $\int_C \vec{F}\cdot\vec{T}\,ds$, $\int_C M\,dx + N\,dy + P\,dz$

$\vec{x} = \vec{r}(t)$ parametric curve

$d\vec{r} = \vec{r}\,'(t)\,dt$ tangent displacement

$ds = \|d\vec{r}\| = \|\vec{r}\,'(t)\|\,dt$ tangent distance (element of arclength)

in $\mathbb{R}^2$, $\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle y', -x'\rangle\,dt$



$d\vec{r} = \vec{T}\,ds = \langle x', y'\rangle\,dt$

$\vec{n}\,ds = \langle y', -x'\rangle\,dt$

$$\underset{\substack{\uparrow \\ \text{graph}}}{\iint_S} f\,dS = \underset{\substack{\uparrow \\ \text{x-y region}}}{\iint_R} f(x,y,g(x,y))\,\underbrace{\sqrt{1+g_x^2+g_y^2}\,dA}_{dS \text{ for a graph } z = g(x,y)}$$

$$\underset{\text{S} \nwarrow \text{graph}}{\iint} \vec{F}\cdot\vec{n}\,dS = \underset{\text{R} \nwarrow \text{x-y region}}{\iint} \vec{F}(x,y,g(x,y))\cdot\langle -g_x, -g_y, 1\rangle\,dx\,dy$$

$dS = \sqrt{1+g_x^2+g_y^2}\,dA$, $\vec{n} = \dfrac{\langle -g_x, -g_y, 1\rangle}{\sqrt{1+g_x^2+g_y^2}}$

$\vec{n}\,dS = \langle -g_x, -g_y, 1\rangle\,dA$

(cancellation!)

or for general parametric surfaces

$$\underset{\substack{\uparrow \\ \text{parameterized surface}}}{\iint_S} f\,dS = \underset{\substack{\uparrow \\ \text{u-v parameterization domain}}}{\iint_R} f(\vec{r}(u,v))\,\|\vec{r}_u\times\vec{r}_v\|\,du\,dv$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

$dS = \|\vec{r}_u\times\vec{r}_v\|\,du\,dv$

$\vec{n} = \dfrac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$

$\vec{n}\,dS = (\vec{r}_u\times\vec{r}_v)\,du\,dv$

- The integral of a constant over a curve, 2-dim'l object, or 3-dim'l object yields that constant times the length, surface area, or volume, respectively!!

Fundamental Theorem(s) of Calculus

$\mathbb{R}^1$: $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

$\mathbb{R}^2$: $\displaystyle\iint_D D_{\vec{u}} f \, dA = \int_{\partial D} f\, \vec{u}\cdot\vec{n}\, ds$

$\mathbb{R}^3$: $\displaystyle\iiint_S D_{\vec{u}} f \, dV = \iint_{\partial D} f\, \vec{u}\cdot\vec{n}\, dS$

$\mathbb{R}^1$ apps :

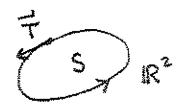
If $\vec{F} = \nabla f$ then $\displaystyle\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)}_{\frac{d}{dt} f(\vec{r}(t))}\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(B) - f(A)$

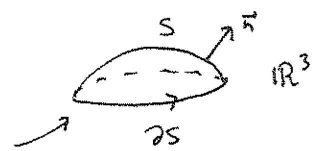
When is $\vec{F}$ a gradient field ∇f?

(curl test)

How to find f in this case.



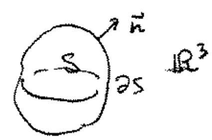
$\mathbb{R}^2$ apps : Green's Thm: $\displaystyle\oint_{\mathcal{C}} \vec{F}\cdot\underbrace{\vec{T}\,ds}_{d\vec{r}} = \iint_S N_x - M_y \, dA$



(& $\mathbb{R}^3$) Stoke's $\displaystyle\oint_{\partial S} \vec{F}\cdot\vec{T}\,ds = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$

div Thm : $\displaystyle\int_{\partial S} \vec{F}\cdot\vec{n}\,ds = \iint_S \vec{\nabla}\cdot\vec{F}\,dA$

$\mathbb{R}^3$ app div Thm: $\displaystyle\iint_{\partial S} \vec{F}\cdot\vec{n}\,dS = \iiint_S \vec{\nabla}\cdot\vec{F}\,dV$



be able to check either side of these FTC's, by computing the integrals.

Sample final exam
illustrative - not inclusive!

The actual spring 2005 final exam is also posted

(It is best to first review and outline the entire course, before trying sample exam questions. Then use sample exams as diagnostic for further study)

1\. a) Sketch the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

b) Show the parametric curve $\vec{r}(t)=\langle 2\cos t, 3\sin t\rangle$ lies on this ellipse

c) Compute $\vec{r}(0), \vec{r}\,'(0), \vec{r}\,''(0)$. Add these vectors to your sketch in (a), at appropriate locations

d) Find the unit vectors $\vec{T}(0)$, $\vec{N}(0)$

e) Our favorite curve equation decomposes acceleration into tangential and normal components:

$$\vec{r}\,''(t) = v'(t)\vec{T} + \kappa v^2 \vec{N}$$

Estimate (with ruler) these tangential and normal components, and compare with the exact values (which could be computed using formulas or with dot product.)

f) for $f(x,y)=\frac{x^2}{4}+\frac{y^2}{9}$, $\vec{r}(t)=\langle 2\cos t, 3\sin t\rangle$, use the multivariable chain rule to compute

$$\frac{d}{dt} f(\vec{r}(t)).$$

Explain why you know the answer will be zero.

2 a) Sketch the surface $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$

b) Find an equation for its tangent plane at the point $(\sqrt{3},0,2)$

3\. Use Green's Theorem with $\vec{F}=\langle -y, x\rangle$, region S the inside of the ellipse in problem 1, and C is boundary curve (parameterized by $\vec{r}(t)$ in problem 1), to prove that the area of the S is 6π.

4\. a) Compute $\int_0^4 \int_{y/2}^{\sqrt{y}} 4x \, dx \, dy$

b) Sketch the region of integration in (a)

c) Express the integral in 4a) as an iterated integral with reversed order of integration, and compute this integral.

5. a) One of these vector fields is a gradient vector field, and one is not. Identify and explain

(i) $\vec{F} = \langle \sin x + e^x \cos y,\ 3y^2 - e^x \sin y \rangle$

(ii) $\vec{F} = \langle -y, x \rangle$

b) Find a function $f(x,y)$ with $\nabla f = \vec{F}$, for the gradient field in (a)

c) For the gradient field in (a), compute

$$\int_C \vec{F} \cdot d\vec{r}$$ where C is any curve from $(0,0)$ to $(\pi/2, \pi/2)$

6. a) If $f(x,y,z) = xy^2/(1+z^2)$, in what (unit) direction is f increasing most rapidly, at the point $(1,1,1)$?

b) Use differentials to approximate $f(1.01, 1.98, 2.03)$.

7. You must build a rectangular shipping crate with volume $60\ ft^3$. Its top costs \$2 / square ft, its sides cost \$1/ft^2, and its bottom costs \$3/ft^2. What dimensions minimize total cost?

8. A uniform wire of density 2 gm/meter is shaped like the semicircle $x^2+y^2=4$, $y \geq 0$. (i.e. its radius is 2 meters)

a) Find the mass of the wire.

b) Find its center of mass.

9. Find the volume of the region that lies inside the sphere $x^2+y^2+z^2=9$ but outside the cylinder $x^2+y^2=4$

10. Consider the vector field $\vec{F} = \langle x, y \rangle$, and the square with vertices $(0,0), (2,0), (0,2), (2,2)$.

a) Compute the flux integral $\int_C \vec{F} \cdot \vec{n}\, ds$ around the sides of the square, by identifying $\vec{n}$ and $\vec{F} \cdot \vec{n}$ along each edge.

b) Use the divergence theorem to recompute the value of the flux integral in (a). (Your answers should agree!)

11. Verify that the $\mathbb{R}^3$ divergence theorem holds for the tetrahedron in the 1st octant with corners at $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ and $\vec{F}(x,y,z) = \langle x, y, z \rangle$.