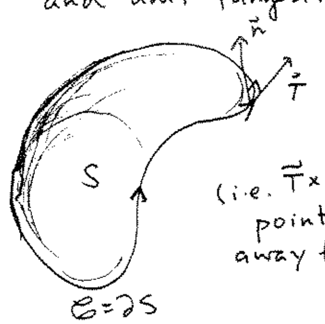
Math 2210-3
Monday April 26

§ 14.7 Stoke's Theorem

This is the final theorem in our FTC zoo. It is the precise interpretation of the statement that curl $\vec{F} = \nabla \times \vec{F}$ measures circulation, and like the divergence theorem it's a key tool in modeling the classical (& modern) partial differential equations of physics, engineering & science.

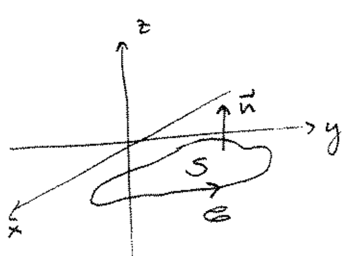
<u>Stokes' Theorem</u> Let S be a two-sided (i.e. not like Möbius strip) surface in $\mathbb{R}^3$, with boundary curve $\mathcal{C}$. Make choice of unit normal $\vec{n}$ to S, and unit tangent vector T on $\mathcal{C}$, so that S is "to the left" as $\mathcal{C}$ is traversed



(i.e. $\vec{T} \times \vec{n}$ points away from surface)

Then $$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

<u>Example 1</u>: Stoke's Theorem contains Green's Theorem as a special case, if you extend $\langle M(x,y), N(x,y) \rangle$ to $\langle M, N, 0 \rangle$ in $\mathbb{R}^3$:



$$\vec{\nabla} \times \langle M, N, 0 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \langle 0, 0, N_x - M_y \rangle$$

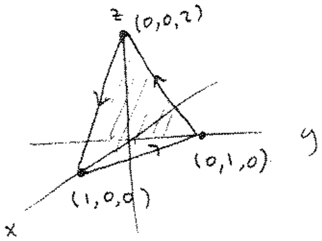
$M = M(x,y)$
$N = N(x,y)$

$\vec{n} = \langle 0, 0, 1 \rangle$

so $$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \iint_S N_x - M_y \, dA \overset{\text{Green's}}{=} \oint_{\partial S} M\,dx + N\,dy$$

Example 2 Use Stoke's Theorem to evaluate $\oint \vec{F} \cdot d\vec{r} = \oint M\,dx + N\,dy + P\,dz$

where $\vec{F} = \langle 2z,\ 8x-3y,\ 3x+y \rangle$ around the indicated triangle

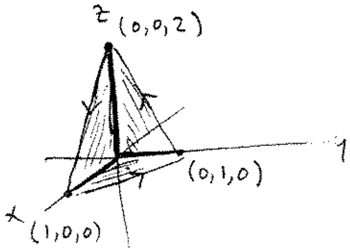


First compute $\vec{\nabla} \times \vec{F}$

2a) Use the triangular piece of plane (notice the equation of this plane is $x + y + \frac{z}{2} = 1$ or $2x + 2y + z = 2$)

$$\iint_{\text{Triangle}} (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS =$$

2b) ~~the~~ You can use any surface which has this oriented triangle as its boundary.
Check that you get the same answer if you use the 3 coordinate triangles!



for xy triangle, $\vec{n} = \langle 0, 0, 1 \rangle$

for yz Δ, $\vec{n} =$

for xz Δ, $\vec{n} =$

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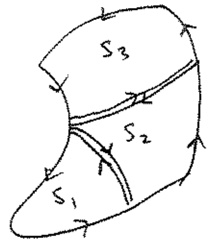
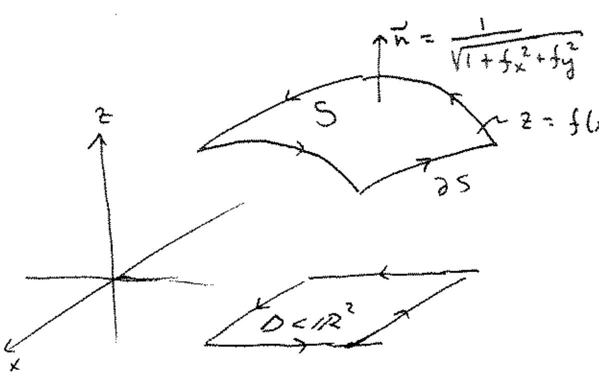
Proof of Stokes'

In fact, the text doesn't think you can handle it, but the proof of Stoke's' theorem is to break general surface S into pieces which are graphs over the various x-y, x-z, or y-z coordinate planes, prove Stoke's for each piece via Greens, and then add up the results to get the global Stoke's theorem.



$S = S_1 \cup S_2 \cup S_3$

Here's a typical x-y graphical piece, and the proof:



$\vec{F} = \langle M, N, P\rangle$

on S, $\vec{F} = \langle M(x,y,f(x,y)), N(x,y,f(x,y)), P(x,y,f(x,y))\rangle$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS = \iint_D \underbrace{\langle P_y - N_z, M_z - P_x, N_x - M_y\rangle}_{\text{all evaluated at } (x,y,f(x,y))} \cdot \underbrace{\frac{1}{\cancel{\sqrt{1+f_x^2+f_y^2}}}\langle -f_x, -f_y, 1\rangle}_{\vec{n}} \underbrace{\cancel{\sqrt{1+f_x^2+f_y^2}}\, dA}_{dS}$$

$$* \quad = \iint_D f_x(N_z - P_y) + f_y(P_x - M_z) + N_x - M_y \quad dA$$

evaluated at $(x,y,f(x,y))$

$$\oint_{\partial S} M\,dx + N\,dy + P\,dz = \oint_{\partial D} M\,dx + N\,dy + P\,(f_x dx + f_y dy)$$

since $z = f(x,y)$

$dz = f_x\,dx + f_y\,dy$

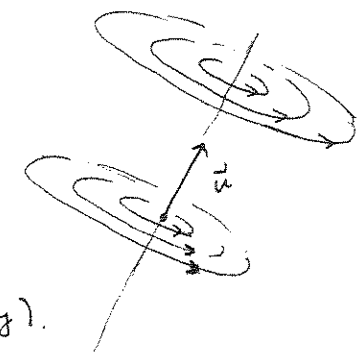
$$= \oint_{\partial D} (M + Pf_x)dx + (N + Pf_y)dy \overset{\text{Green's}}{=} \iint_D (N + Pf_y)_x - (M + Pf_x)_y \, dA$$

Chain rule + prod rule

$$= \iint_D N_x + N_z f_x + (P_x + \cancel{P_z f_x}) f_y + \cancel{P f_{yx}} - (M_y + M_z f_y + (\cancel{(P_y + P_z f_y)}^{f_x} + \cancel{P f_{xy}}) \, dA$$

$$= \iint_D f_x(N_z - P_y) + f_y(P_x - M_z) + N_x - M_y = * \text{ !!! } \blacksquare$$

Example 3 Let $\vec{u}$ be a unit vector, $\vec{u} = \langle a, b, c\rangle$
Let $k \in \mathbb{R}$



Let $\vec{F}(x,y,z) = k\vec{u} \times \langle x, y, z\rangle$

be the "strength" k right-handed rotation velocity field (actually, k is the angular velocity).

3a) Show $\vec{F} = k\langle bz - cy, cx - az, ay - bx\rangle$

3b) Show $\vec{\nabla} \times \vec{F} = 2k\vec{u}$ recovers the axis of rotation.

3c) Let $\mathcal{C}$ be any curve which goes around the axis once, in the indicated direction. Suppose that $\mathcal{C}$ projects onto a plane $\perp$ to the axis $\vec{u}$, as the boundary of a domain D of area area(D). Show

$$\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 2k\,(\text{area}(D))$$

Hint: Use the surface consisting of D and the part of the $\vec{u}$-direction "cylinder" between ∂D and $\mathcal{C}$