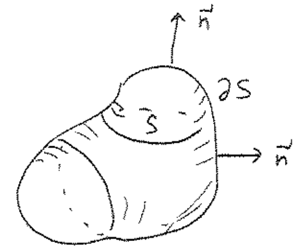
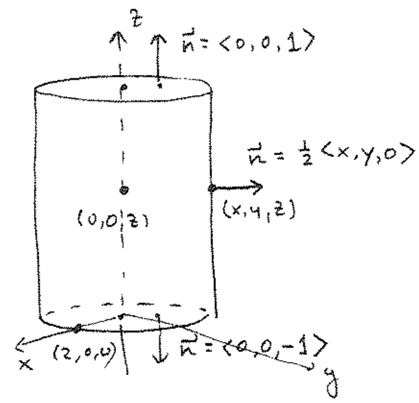
Math 2210
Friday April 23
§14.6 : the $\mathbb{R}^3$ divergence theorem (Gauss' Theorem)

In §14.5 we discussed surface integrals and flux integrals.
The $\mathbb{R}^3$ divergence theorem says that if $\vec{F} = \langle M, N, P \rangle$ is a continuously differentiable vector field in a region S, bounded by a surface ∂S with unit exterior normal $\vec{n}$, then

$$\boxed{\iint_{\partial S} (\vec{F} \cdot \vec{n})\, dS = \iiint_{S} (\operatorname{div} \vec{F})\, dV}$$



Exercise 1 : Let S be the solid cylinder of rad 2 and height 6
$x^2 + y^2 \le 4$
$0 \le z \le 6$
Let $\vec{F} = \langle x, y, z \rangle$
Verify the divergence theorem in this case.
Picture at right may help.



In practice the div thm can be used to compute one integral's value by computing the other's. It's also the key theorem for deriving many important partial differential equations. (See page 5...)

The Basic (any variable) FTC.

(and how it implies div thm in any $\mathbb{R}^n$).

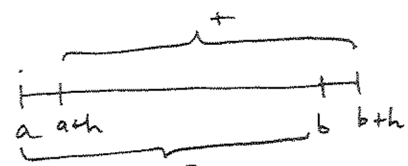
$$\boxed{n=1 \qquad \int_a^b F'(x)\,dx = F(b)-F(a)}$$

usually we write $f(x)$ for $F'(x)$

nifty way to see this: let $h>0$ small

$$\int_a^b F'(x)\,dx \approx \int_a^b \frac{F(x+h)-F(x)}{h}\,dx$$

$$= \frac{1}{h}\Big[\underbrace{\int_a^b F(x+h)\,dx}_{\substack{u=x+h \\ du=dx}} - \int_a^b F(x)\,dx$$



$$= \frac{1}{h}\Big[\int_{a+h}^{b+h} F(u)\,du - \int_a^b F(x)\,dx$$

notice $\int_{a+h}^{b} F(x)\,dx$ cancels

$$= \frac{1}{h}\Big[\int_b^{b+h} F(x)\,dx - \int_a^{a+h} F(x)\,dx\Big]$$

$$\approx F(b)-F(a). \quad ∎$$

It's possible to turn this into a proof, by letting $h \to 0$.

$n=2$ Let $\vec{u}$ be any unit vector.
Let S be a region bounded by a curve $\mathcal{C}$, with unit normal (exterior) $\vec{n}$
Then if f is differentiable on S,

$$\boxed{n=2 \qquad \iint_S D_{\vec{u}} f\,dA = \int_{\mathcal{C}} f\,\vec{u}\cdot\vec{n}\,ds}$$

$n=3$: Let $\vec{u}$ be any unit vector.
Let $S \subset \mathbb{R}^3$ be a region bounded by a surface ∂S, with unit exterior normal $\vec{n}$
Then for differentiable f,

$$\boxed{n=3 \qquad \iiint_S D_{\vec{u}} f\,dV = \iint_{\partial S} f\,\vec{u}\cdot\vec{n}\,dS}$$

In fact, the analogous theorem is true in any space dimension, and the case $n=1$ is included, since $F'(x) = D_u f$ for $u=+1$.

The $n=2$ picture indicates why this FTC is true in all space dimensions

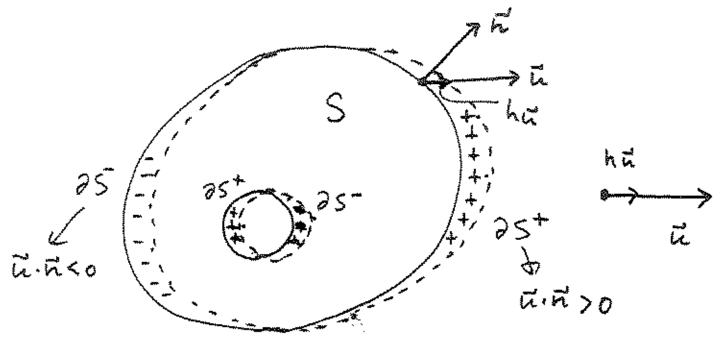
$$\boxed{\iint_S D_{\vec u} f \, dA} \approx \iint_S \frac{f(\vec x + h\vec u) - f(\vec x)}{h} \, dA$$

$$= \frac{1}{h}\left[\iint_S f(\vec x + h\vec u)\, dA - \iint_S f(\vec x)\, dA\right.$$

translation COV $\vec z = \vec x + h\vec u$ leaves dA unchanged.

$$* \qquad = \frac{1}{h}\left[\iint_{\underbrace{S+h\vec u}_{\substack{S \text{ translated} \\ \text{by } h\vec u}}} f(\vec z)\, dA - \iint_S f(\vec x)\, dA\right]$$

integral's away from ∂S cancel out,
leaving S_h^+ with + contribution,
S_h^- with − contribution



$$* = \frac{1}{h}\iint_{S_h^+} f(\vec x)\, \underbrace{dA}_{\approx} - \frac{1}{h}\iint_{S_h^-} f(\vec x)\, \underbrace{dA}_{\approx}$$

$$\approx \frac{1}{\cancel{h}}\int_{\partial S^+} f \, \overbrace{\cancel{h}\vec u \cdot \vec n \, ds} - \frac{1}{\cancel{h}}\int_{\partial S^-} f \, \overbrace{(-h\vec u \cdot \vec n)\, ds}$$

$$\boxed{= \int_{\partial S} f \, \vec u \cdot \vec n \, ds} \quad \blacksquare$$

One can take $\lim_{h\to 0}$ and create an actual proof, although it's technically difficult so textbooks use iterated integrals and decomposition ideas like we did for Green's Theorem.

for $n=3$, do you see how the argument would go?

$n=2 \quad n=3$

§14.4, 14.6

FTC $\Longrightarrow$ divergence Theorem

Monday

$n=2$ divergence $\Longrightarrow$ Green's. $\Longrightarrow$ Stoke's

14.4   14.7

<u>$n=2$</u> : let $\vec{F} = \langle M, N \rangle$

FTC, $\vec{u} = \langle 1, 0 \rangle$: $$\iint_S \frac{\partial M}{\partial x}\, dA = \int_{\mathscr{C}} M \,\langle 1,0\rangle \cdot \vec{n}\, ds$$

$\vec{u} = \langle 0, 1 \rangle$: $$\iint_S \frac{\partial N}{\partial y}\, dA = \int_{\mathscr{C}} N \,\langle 0,1\rangle \cdot \vec{n}\, ds$$

<u>add</u> : $$\boxed{\iint_S \vec{\nabla} \cdot \vec{F}\, dA = \int_{\mathscr{C}} \langle M, N \rangle \cdot \vec{n}\, ds} \quad \blacksquare$$

$n=3$: let $\vec{F} = \langle M, N, P \rangle$

FTC, $\vec{u} = \hat{\imath}$: $$\iiint_S M_x\, dV = \iint_{\partial S} M \,\langle 1,0,0\rangle \cdot \vec{n}\, dS = \iint_{\partial S} \langle M,0,0\rangle \cdot \vec{n}\, dS$$

$\vec{u} = \hat{\jmath}$: $$\iiint_S N_y\, dV = \iint_{\partial S} N \,\langle 0,1,0\rangle \cdot \vec{n}\, dS$$

$\vec{u} = \hat{k}$: $$\iiint_S P_z\, dV = \iint_{\partial S} P \,\langle 0,0,1\rangle \cdot \vec{n}\, dS$$

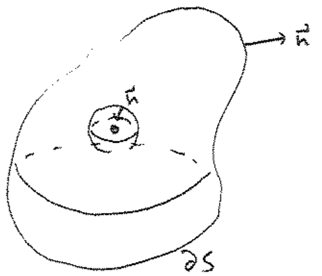
<u>add</u> : $$\boxed{\iiint_S \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial S} (\vec{F} \cdot \vec{n})\, dS}$$

<u>etc.</u>

<u>Exercise 2</u> : Let S be a solid region containing a point mass M (or a point charge, actually) in its interior, with corresponding inverse square field

$$\vec{F} = -cM\frac{\vec{r}}{\|\vec{r}\|^3}$$

Show that the flux $\displaystyle\iint_{\partial S} \vec{F}\cdot\vec{n}\,dS = -4\pi c M$ regardless of the shape of S.



Hint: Let W be the solid obtained by deleting a ball of sufficiently small radius a centered at the mass. Apply the divergence theorem on W

<u>Remark</u> Since force fields are additive we can ~~use~~ extend the same argument to deduce that if S contains k point masses $M_1, M_2 .. M_k$ at varying points, then

$$\iint_{\partial S} \vec{F}\cdot\vec{n}\,dS = -4\pi c\,(M_1 + M_2 + \dots + M_k)$$

Gauss' Law (for electric charge, usually).

Then by a Riemann sum/limit argument you can extend this theorem to a continuously distributed mass (or charge) density $\rho(x,y,z)$

$$\begin{array}{ccc} \displaystyle\iint_{\partial S} \vec{F}\cdot\vec{n}\,dS & = & -4\pi c M \\ \| & & \| \\ \displaystyle\iiint_S \operatorname{div}\vec{F}\,dV & = & \displaystyle -4\pi c \iiint_S \rho\,dV \end{array}$$

Leads to one of $\boxed{\text{Maxwell's laws for electromagnetism}}$: $\boxed{\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}}$

the other 3 are:

$$\operatorname{div}\vec{B} = 0$$

$$\operatorname{curl}\vec{E} = -\frac{\partial B}{\partial t}$$

$$\operatorname{curl}\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$\vec{E}$ = electric field.
$\vec{B}$ = magnetic field
$\vec{J}$ = current density.