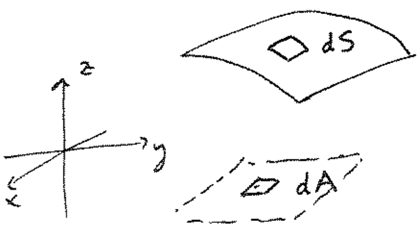
Math 2210-3
Wednesday April 21

How for Wed 4/28
14.5 3, 7, 9, 10, 17, 19, 20
14.6 1, 7, 15, 16, 19
14.7 3, 7, 8, 9, 19

§14.5 Surface Integrals

Recall for a graph $z = f(x,y)$



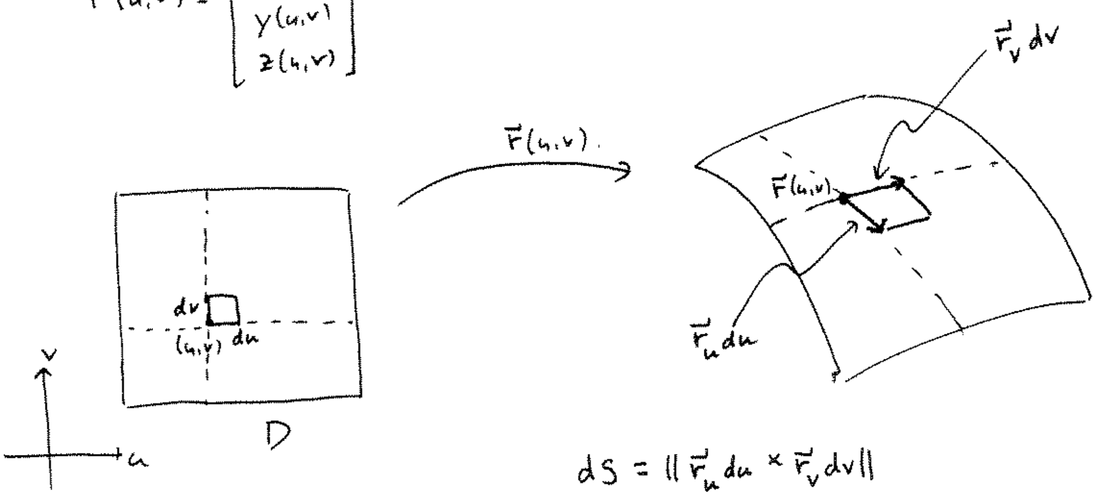
$$\boxed{dS = \sqrt{1+f_x^2+f_y^2}\; dA}$$

area expansion factor

this was a special case of the area element for parametric surfaces, which we discussed March 29:

$$\vec{F} : \underset{\mathbb{R}^2}{\underset{\cap}{D}} \longrightarrow \mathbb{R}^3$$

$$\vec{F}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$



$$dS = \|\vec{F}_u\,du \times \vec{F}_v\,dv\|$$

$$\boxed{dS = \|\vec{F}_u \times \vec{F}_v\|\; du\,dv}$$

check: the graph formula for dS follows from $\vec{F}(x,y) = \langle x, y, f(x,y)\rangle$ and the parametric one.

check for later: the right-handed unit normal vector is

$$\vec{n} = \frac{\vec{F}_u \times \vec{F}_v}{\|\vec{F}_u \times \vec{F}_v\|}$$

for $z = f(x,y)$, $\vec{n} = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{1+f_x^2+f_y^2}}$

In Chapter 13 we used page 1 formulas to compute surface area.

We can also use them to compute surface integrals:

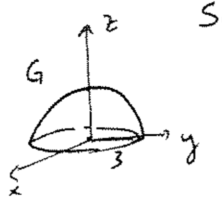
Def. Let g be a scalar function defined on a surface G.

Then if G is parameterized by $F(u,v)$, from a domain D,

- $$\iint_G g\, dS = \iint_D g(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$

- If G is a graph $z = f(x,y)$, above domain D
$$\iint_G g\, dS = \iint_D g(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}\, dA$$

Exercise 1 Let G be the upper hemisphere of radius 3 centered at the origin (3 meters)



Suppose this shell has density $\delta(x,y,z) = z$ gm/m^2 at height z

Find the mass of this hemispherical shell,

$$\iint_G \delta\, dS$$

ans 12π gm

(surface) flux integrals

If $\vec{F}$ is a vector field and G has a unit normal $\vec{n}$ then the flux integral

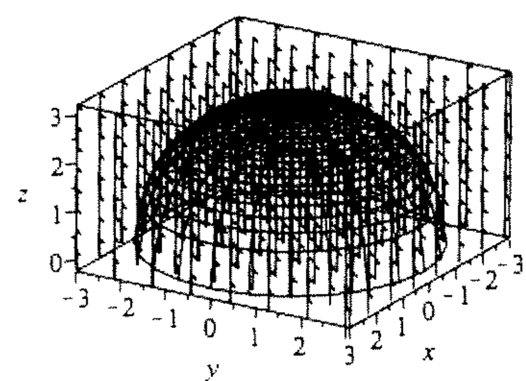
- $$\iint_G \vec{F}\cdot\vec{n}\,dS = \iint_D \left(\vec{F}(\vec{r}(u,v))\cdot\frac{(\vec{r}_u\times\vec{r}_v)}{\cancel{\|\vec{r}_u\times\vec{r}_v\|}}\right)\cancel{\|\vec{r}_u\times\vec{r}_v\|}\,du\,dv$$
$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

graph:
- $$\iint_G \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}(x,y,f(x,y))\cdot\frac{\langle -f_x,-f_y,1\rangle}{\cancel{\sqrt{1+f_x^2+f_y^2}}}\,\cancel{\sqrt{1+f_x^2+f_y^2}}\,dx\,dy$$

Exercise 2 Let $\vec{F} = \langle 0,0,5\rangle$

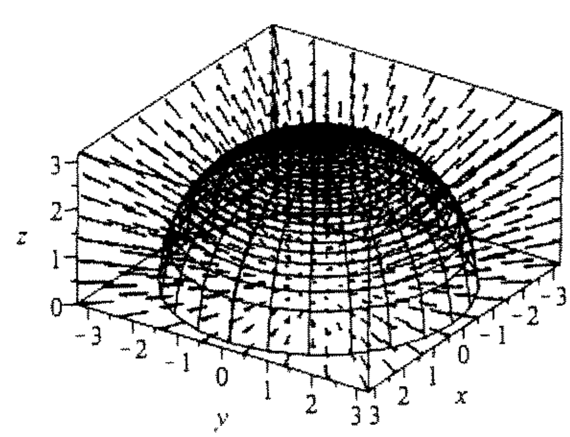
Let G be the upper hemisphere in exercise 1, with upper unit normal $\vec{n}$

Compute
$$\iint_G (\vec{F}\cdot\vec{n})\,dS$$



Exercise 3 Same G. Now $\vec{F} = \langle x,y,z\rangle$

Compute $\iint_G \vec{F}\cdot\vec{n}\,dS$ without computing an integral

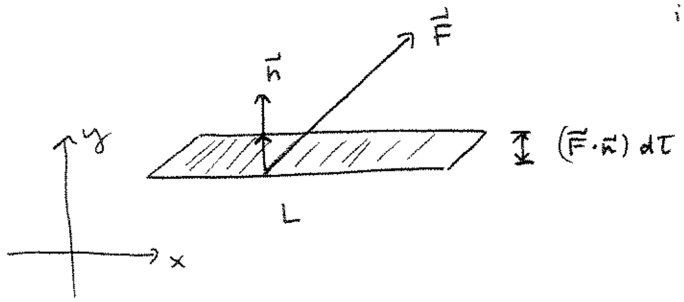


flux integrals interpretation

$\mathbb{R}^2$: Case I   Constant field $\vec{F}$, particle velocity field.

Line segment of length L, unit normal $\vec{n}$

in time $d\tau$ particles move $(d\tau)\vec{F}$

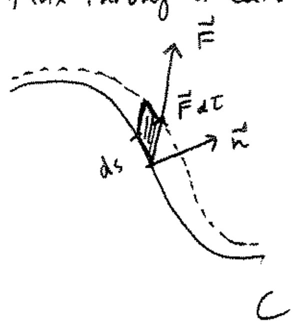


an area $d(\text{Area}) = L\,(\vec{F}\cdot\vec{n})\,d\tau$ passes through

$$\boxed{\frac{d\,(\text{Area})}{dt} = L(\vec{F}\cdot\vec{n})}$$

- the flux through the segment, $L\,\vec{F}\cdot\vec{n}$ is how fast area is passing through.

Case II   flux through a curve, varying $\vec{F}$

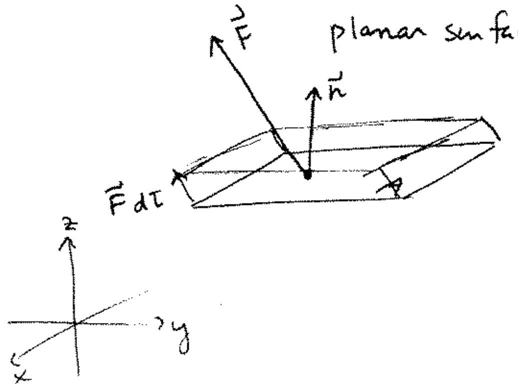


$$d(\text{Area}) = \int_{\mathcal{C}} (d\tau)\vec{F}\cdot\vec{n}\,ds = (d\tau)\int_{\mathcal{C}} \vec{F}\cdot\vec{n}\,ds$$

$$\boxed{\frac{d\,(\text{Area})}{d\tau} = \int_{\mathcal{C}} \vec{F}\cdot\vec{n}}$$

$\mathbb{R}^3$: Case I: Constant $\vec{F}$ velocity field.

planar surface, area A



$$d\,(\text{Vol}) = A(\underbrace{\vec{F}\,d\tau\cdot\vec{n}}_{\text{height}}) = (A\,\vec{F}\cdot\vec{n})\,d\tau$$

$$\boxed{\frac{d\,(\text{Vol})}{d\tau} = A\,(\vec{F}\cdot\vec{n})}$$

Case II: Surface G, vector field $\vec{F}$

unit normal $\vec{n}$

$$\boxed{\frac{d\,\text{Vol}}{dt} = \iint_G (\vec{F}\cdot\vec{n})\,dS}$$

Exercise 4

Compute $\iint_G \vec{F}\cdot\vec{n}\,dS$

for the cylinder G: $x^2+y^2=4$, $0\le z\le 6$, $\vec{n}$ = outward unit normal

$\vec{F} = \langle x, y, z\rangle$

a) without computing any integrals

b) using

$\vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$ and page 3 formula.