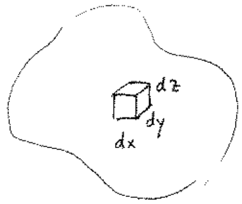
Math 2210-3
Friday April 2

§13.8: Triple integrals in spherical & cylindrical coords
13.9: general COV.

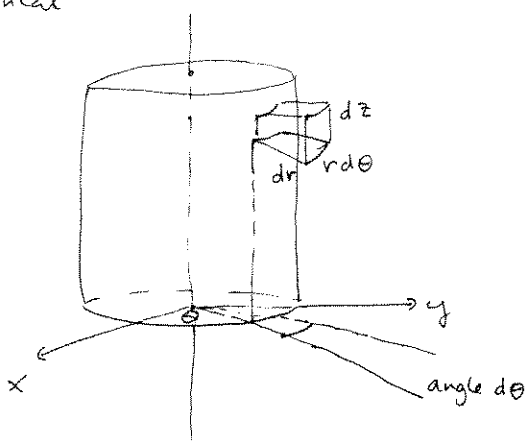
rectangular:



$dV = (dx)(dy)(dz)$

but you can partition a region using other coord systems, just like we used polar coords for double integrals

cylindrical coords



from polar coords
↓
$dV = (dA)\,dz$

$$\boxed{dV = r(dr)(d\theta)(dz)}$$

so $\iiint_R f(x,y,z)\,dV = \iiint f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$

R specified in cylind. coords

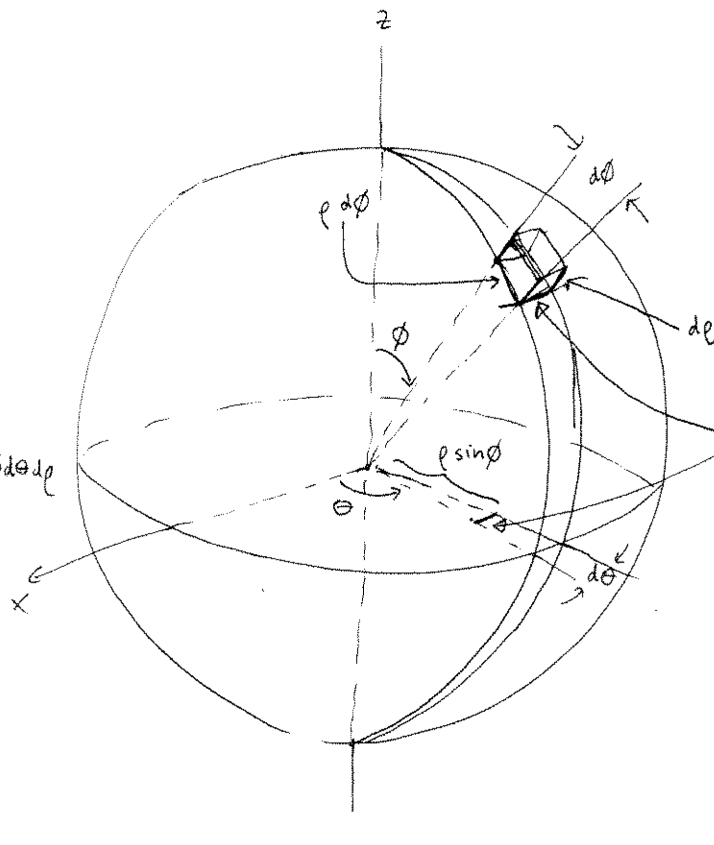
spherical coords

So

$\iiint_R f(x,y,z)\,dV$

$= \iiint f\begin{pmatrix}\rho\cos\theta\sin\phi\\ \rho\sin\theta\sin\phi\\ \rho\cos\phi\end{pmatrix}\rho^2\sin\phi\; d\phi\, d\theta\, d\rho$

R specified in spherical coords

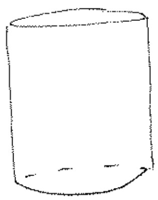


$? = \rho\sin\phi\; d\theta$

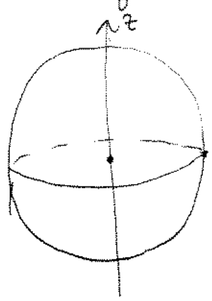
$$\boxed{dV = \rho^2\sin\phi\; d\phi\, d\theta\, d\rho}$$

Examples

Volume of cylinder of radius a and height h:

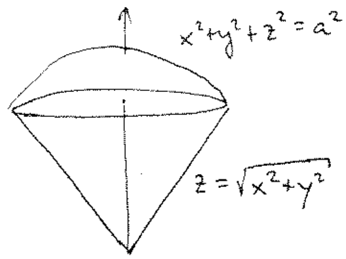


Volume of ball of radius a



Example : How much ice cream in a full ice cream cone?

region inside $x^2+y^2+z^2=a^2$
and the ice-cream cone $z=\sqrt{x^2+y^2}$



Find volume & centroid ($\delta \equiv 1$). (in case it starts rotating when you drop it.)

ans $Vol = \frac{2}{3}\pi a^3 \left(1-\frac{1}{\sqrt{2}}\right)$.

$\bar{z} = \frac{3}{8(2-\sqrt{2})} a$

```
> Vol:=int(int(int(rho^2*sin(phi),phi=0..Pi/4),rho=0..a),theta=0..2*
  Pi);
```

$$Vol := \frac{2\left(-\frac{\sqrt{2}}{2}+1\right)a^3\pi}{3}$$

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> Mxy:=int(int(int(rho*cos(phi)*rho^2*sin(phi),phi=0..Pi/4),rho=0..a
  ),theta=0..2*Pi);
```

$$Mxy := \frac{a^4\pi}{8}$$

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> zbar:=Mxy/Vol;
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$$zbar := \frac{3a}{16\left(-\frac{\sqrt{2}}{2}+1\right)}$$

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>
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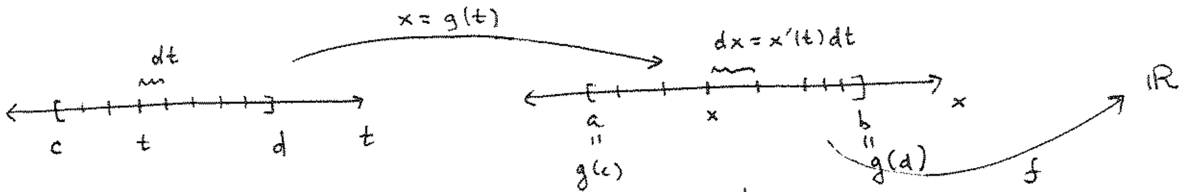
§13.9 Big picture for change of variables includes Calc I :



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compute $\int_a^b f(x)\,dx$

by partitioning the t-interval, and transforming that into a partition of the x-interval, via the function g.

i.e. $\boxed{\begin{matrix} x = g(t) \\ dx = g'(t)\,dt \end{matrix}}$ $\quad \Delta x_i \approx g'(t_i)\Delta t_i$

single integral COV

$$\boxed{\int_{t=g^{-1}(a)}^{g^{-1}(b)} f(g(t))\,g'(t)\,dt = \int_a^b f(x)\,dx}$$

we actually checked this with FTC, since $F(b)-F(a)$
$= F(g(g^{-1}(b))) - F(g(g^{-1}(a)))$

and used this method to compute the left-side integral by the right-side one.

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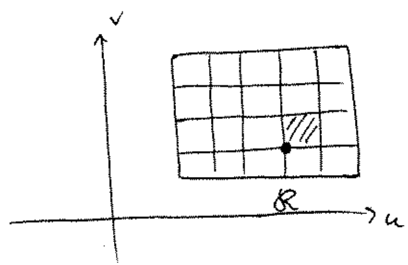
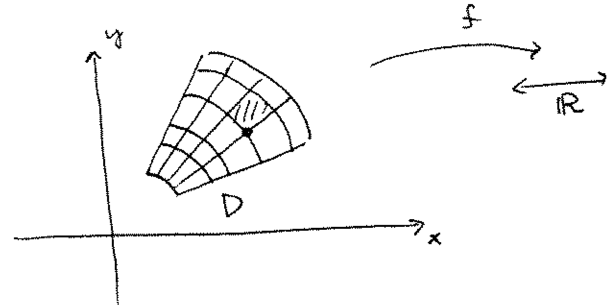


$\xrightarrow{\vec r}$

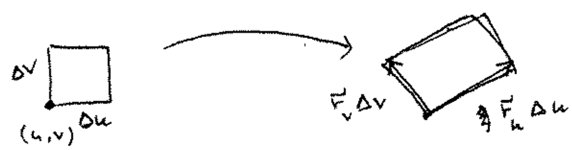
$\vec r(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \end{bmatrix}$



partition D by transforming partitions of $\mathcal{R}$

magnify shaded pieces

e.g. polar coords $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$



$$\Delta A \approx \left|\det\left[\vec r_u \Delta u \,\middle|\, \vec r_v \Delta v\right]\right|$$

$$= \left|\det\begin{bmatrix} x_u \Delta u & x_v \Delta v \\ y_u \Delta u & y_v \Delta v \end{bmatrix}\right| = \underbrace{\left|\det\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\right|}_{\text{write } \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \text{ "Jacobian"}} \Delta u \Delta v$$

area expansion factor

$$\boxed{dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

double integral COV

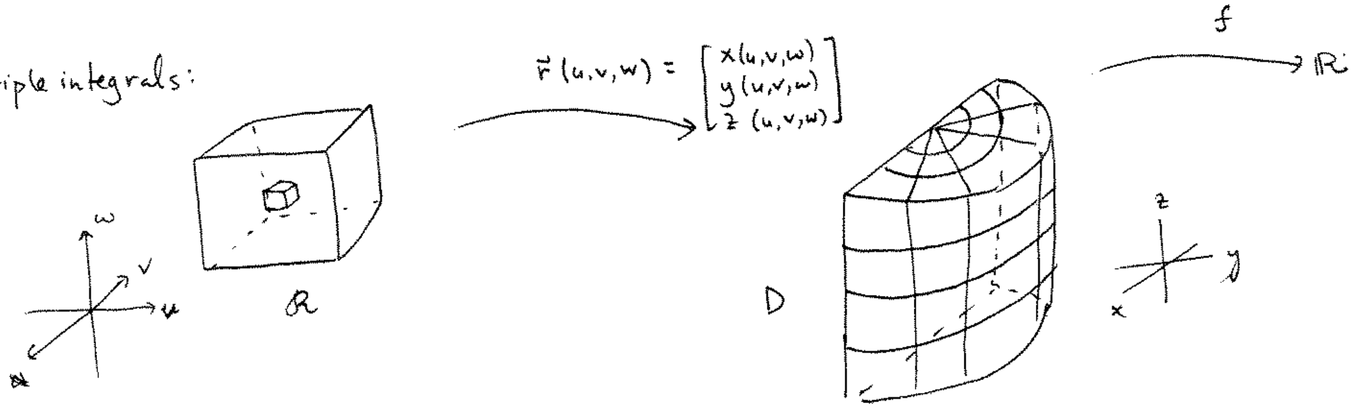
$$\boxed{\iint_{\mathcal{R}} f(\vec r(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_D f(x,y)\,dA}$$

<u>Exercise</u>: Check that the general 2-d change of variables formula yields $dA = r\,dr\,d\theta$

for polar coord COV, $x = r\cos\theta$, $y = r\sin\theta$.

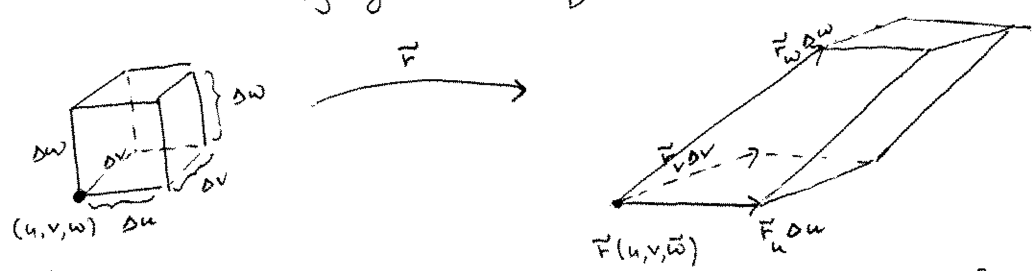
these means you need to compute the Jacobian matrix and determinant.

triple integrals:



$$\iiint_D f(x,y,z)\,dV$$

magnify!



triple integral COV

$$\Delta V \approx \text{abs}\left(\det\left[\vec{F}_u \Delta u \,\middle|\, \vec{F}_v \Delta v \,\middle|\, \vec{F}_w \Delta w\right]\right)$$

$$\boxed{\iiint_R f\left(\vec{F}(u,v,w)\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \iiint_D f(x,y,z)\,dV}$$

$$dV = \left(\left|\begin{matrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{matrix}\right|\right) du\,dv\,dw$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \quad \text{volume expansion factor}$$

Exercise

Compute the Jacobian determinant to show that this general formula produces

$$dV = r\,dr\,d\theta\,dz \qquad \text{cylindrical}$$
$$dV = \rho^2 \sin\phi\; d\rho\, d\theta\, d\phi. \qquad \text{spherical.}$$

try one!

(In hw you get to use different COV too.)