Math 2210-3
Monday April 19

14.4 Green's Theorem and $\mathbb{R}^2$ divergence theorem; meaning of curl and div revisited.

<u>Green's Theorem</u> Let $\vec{F} = \langle M, N \rangle$ be a continuously differentiable vector field defined on a region S. Let C be its boundary, "∂S", piecewise smooth, oriented with S on the left as you traverse ∂S. (Counterclockwise if S has no holes.

Then $$\oint_{\partial S} M\,dx + N\,dy = \iint_S \underbrace{N_x - M_y}_{\text{curl}(\vec{F})}\, dA$$

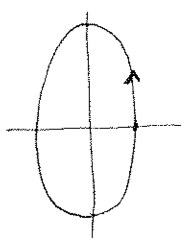
- go over proof, page 2 (= page 5 Friday).

After going through that material, try using Green's thm to evaluate <u>Example</u>:

$$\oint_C (x^2 + 2y + xy)\,dx + (4x - 3y^2)\,dy \qquad = \iint_S (N_x - M_y)\,dA$$

where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$\vec{r}(t) = \langle 2\cos t, 3\sin t \rangle$
$\vec{r}\,'(t) = \langle -2\sin t, 3\cos t \rangle$



direct Maple check: (would also see directly)

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> Int((4*cos(t)^2+6*sin(t)+6*cos(t)*sin(t))*(-2*sin(t)
  )+(4*2*cos(t)-3*(9*sin(t)^2))*3*cos(t),t=0..2*Pi);
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$$\int_0^{2\pi} -2\,(4\cos(t)^2 + 6\sin(t) + 6\cos(t)\sin(t))\sin(t) + 3\,(8\cos(t) - 27\sin(t)^2)\cos(t)\,dt$$

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> int((4*cos(t)^2+6*sin(t)+6*cos(t)*sin(t))*(-2*sin(t)
  )+(4*2*cos(t)-3*(9*sin(t)^2))*3*cos(t),t=0..2*Pi);
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$$12\pi$$

Friday ~~5~~
Monday 2

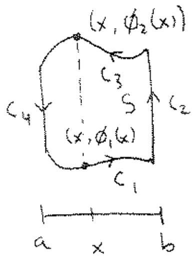
Proof of Green's Thm

1 $$\iint_S M_y \, dA = \oint_{\partial S} -M \, dx$$

2 $$\iint_S N_x \, dA = \oint_{\partial S} N \, dy$$

3 : 2 - 1 $\Rightarrow$ $$\iint_S N_x - M_y \, dA = \int_{\partial S} M\,dx + N\,dy$$ ∎ (After 1 2)

1 If S is y-simple:



$$\iint_S M_y \, dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} M_y \, dy \right) dx$$

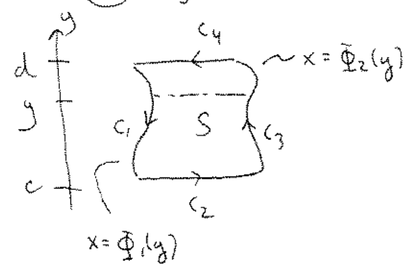
$$= \int_a^b M(x,y) \Big]_{\phi_1(x)}^{\phi_2(x)} dx$$

$$= \int_a^b M(x, \phi_2(x)) \, dx - \int_a^b M(x, \phi_1(x)) \, dx$$

$$= -\int_{C_3} M \, dx - \int_{C_1} M \, dx$$

$$= -\oint_C M(x,y) \, dx$$ because $dx \equiv 0$ on C_4 & C_2!

2 If S is x-simple



$$\iint_S N_x \, dA = \int_c^d \int_{\Phi_1(y)}^{\Phi_2(y)} N_x(x,y) \, dx \quad dy$$

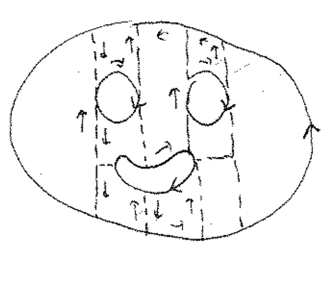
$$= \int_c^d N(x,y) \Big]_{x=\Phi_1(y)}^{x=\Phi_2(y)} dy$$

$$= \int_c^d N(\Phi_2(y), y) \, dy - \int_c^d N(\Phi_1(y), y) \, dy$$

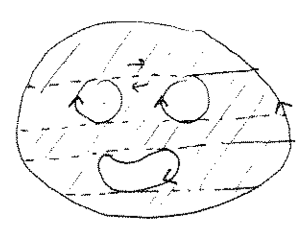
$$= \int_{C_3} N(x,y) \, dy + \int_{C_1} N(x,y) \, dy$$

$$= \oint_C N(x,y) \, dy$$

For general S, decompose into vertically-simple subdomains and add up Green's identity over all of them. Line integrals on inside cancel out, giving result!



General S:

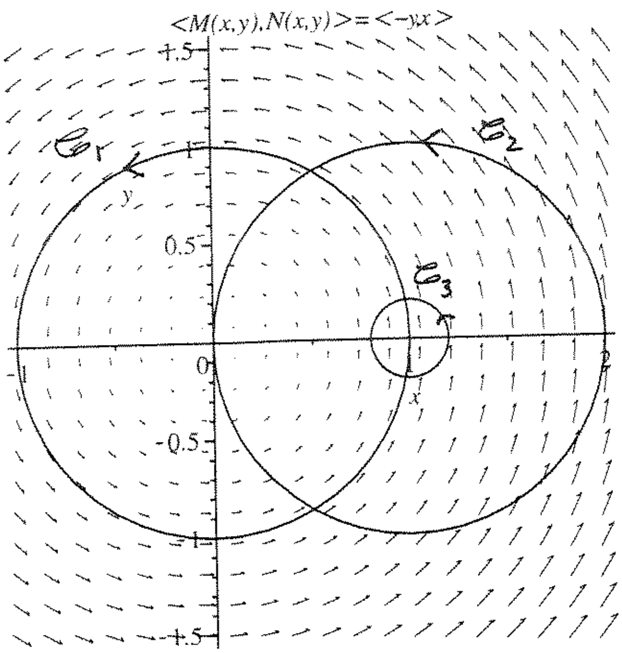


∎

<u>Green's Theorem</u> makes precise, in $\mathbb{R}^2$, how the curl of a vector field measures "circulation"

$$\underbrace{\oint_{\mathcal{C}} \vec{F}(\vec{r})\cdot d\vec{r} = \oint_{\mathcal{C}} \vec{F}\cdot\vec{T}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt}_{\text{"circulation" around } \mathcal{C}\text{, enclosing } S} \overset{\text{Green!}}{=} \underbrace{\iint_S N_x - M_y \, dA}_{\text{integral of scalar curl inside } \mathcal{C}.}$$

Discuss $\boxed{\text{circulation integrals}}$ for
$\mathcal{C}_1 : x^2+y^2=1$
$\mathcal{C}_2 : (x-1)^2+y^2=1$
$\mathcal{C}_3 : (x-1)^2+y^2=.04$

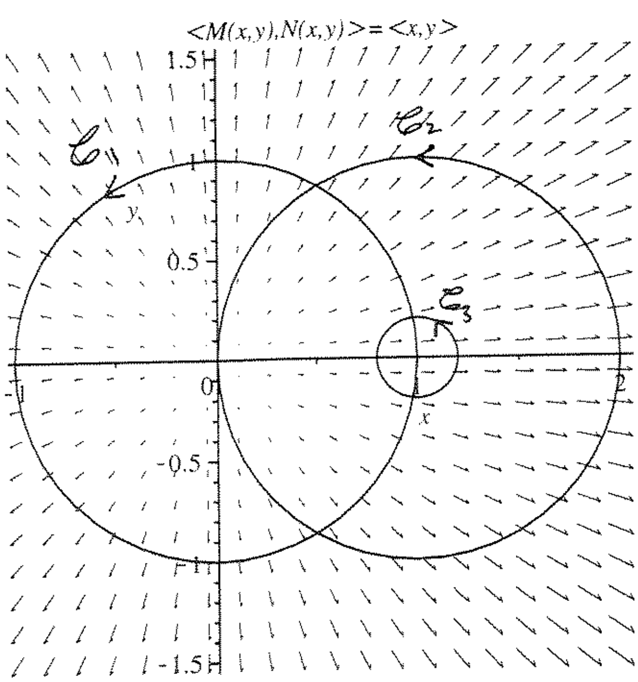


$\vec{F} = \langle -y, x\rangle$

$$\int_{\mathcal{C}_1} \vec{F}\cdot\vec{T}\,ds =$$

$$\int_{\mathcal{C}_2} \vec{F}\cdot\vec{T}\,ds =$$

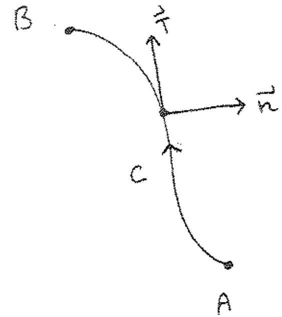
$$\int_{\mathcal{C}_3} \vec{F}\cdot\vec{T}\,ds =$$



$\vec{F} = \langle x, y\rangle$, now what are the three circulation integrals!

Flux integrals

$\vec{F} = \langle M, N \rangle$ vector field.



flux integral:

$$\int_C \vec{F} \cdot \vec{n}\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\langle y'(t), -x'(t) \rangle}{\cancel{\|\vec{r}'(t)\|}} \cancel{\|\vec{r}'(t)\|}\, dt$$

curve $\vec{x} = \vec{r}(t)$, $a \le t \le b$

$\vec{T}$ = unit tangent vector;

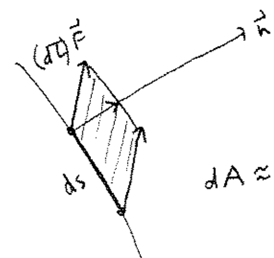
$\vec{T} = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{\langle x', y' \rangle}{\|\vec{r}'\|}$, $\quad ds = \|\vec{r}'(t)\| dt$

$\vec{n}$ = unit normal (oriented as shown)

$= \dfrac{\langle y', -x' \rangle}{\|\vec{r}'\|}$

called "flux" because one interpretation considers $\vec{F}$ as a velocity vector field of moving fluid film (2-d), and $\int_C \vec{F} \cdot \vec{n}\, ds$ measures how fast the fluid is moving across C (area/time):

magnify: τ = time, $d\tau$ an increment



$dA \approx ds\,(d\tau\, \vec{F} \cdot \vec{n})$

$$A(d\tau) - A(0) \approx d\tau \int_C \vec{F} \cdot \vec{n}\, ds$$

$$\frac{dA}{d\tau} = \int_C \vec{F} \cdot \vec{n}\, ds.$$

Green's thm lets us evaluate flux integrals around closed loops:

$n=2$
Div thm:

$$\iint_S \operatorname{div} \vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, ds$$

proof (see above)

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \langle M, N \rangle \cdot \langle y', -x' \rangle\, dt$$

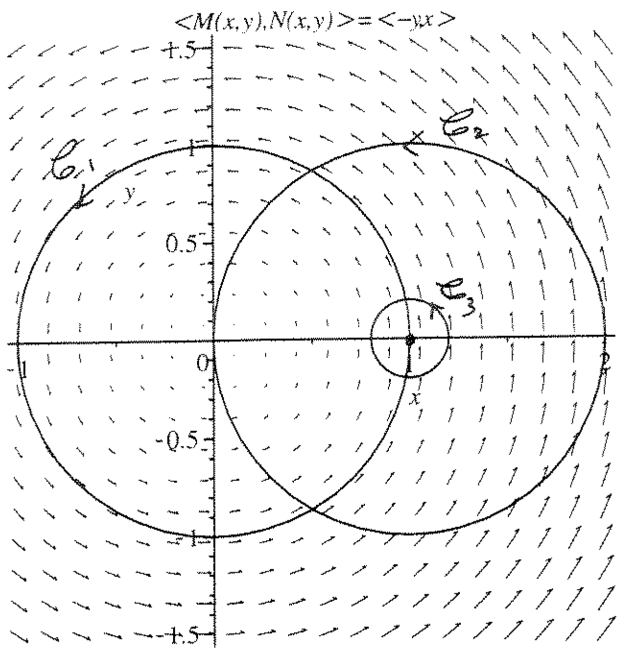
$$= \int_a^b \langle -N, M \rangle \cdot \langle x', y' \rangle\, dt$$

$$= \oint_C -N\,dx + M\,dy = \iint_S M_x - -N_y\, dA = \iint_S M_x + N_y\, dA = \iint_S \operatorname{div} \vec{F}\, dA$$

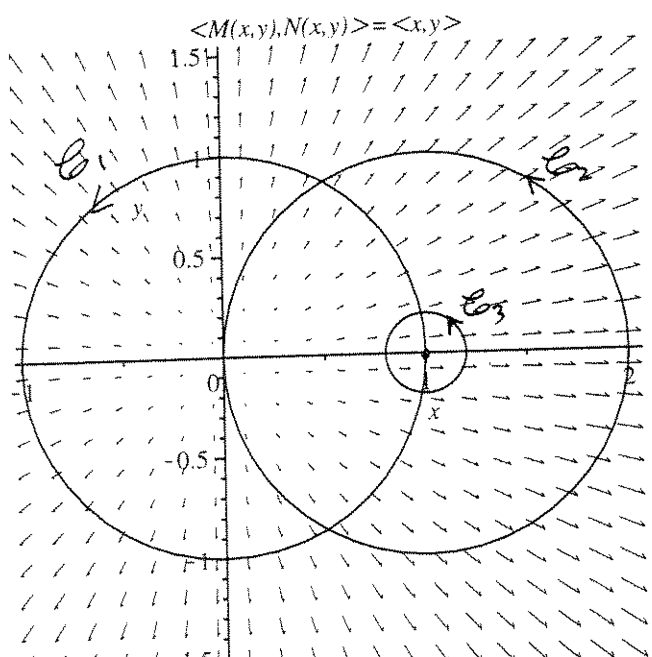
the $n=2$ divergence theorem makes precise how the $\operatorname{div}\vec{F} = \operatorname{div}\langle M,N\rangle = M_x + N_y$ measures "diverging"ness of vector fields

$$\underbrace{\oint_{\partial S} \vec{F}\cdot\vec{n}\,ds}_{\text{total flux}} = \iint_S \operatorname{div}\vec{F}\,dA.$$

Discuss $\boxed{\text{flux integrals}}$ for same $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$ & $\vec{F}$ as page 3!



$\vec{F} = \langle -y, x\rangle$



$\vec{F} = \langle x, y\rangle$