Math 2210-3
Friday April 16

§14.3-14.4 (We'll use parts of these notes on Monday too.)

↑ path independence; ↖ Green's theorem

$$\int_{C} g(\vec{x})\, ds = \quad \text{(scalar fcn)}$$

$$\int_{C} \vec{F}(\vec{r}) \cdot d\vec{r} =$$

On Wednesday we showed

- If $\vec{F} = \nabla f$, C a path from A to B, then
$$\int_{C} \vec{F} \cdot d\vec{r} = f(B) - f(A).$$
- We've verified several times that in order for $\vec{F}$ to be a gradient field, it's necessary for $\vec{\nabla} \times \vec{F} = \vec{0}$. $(N_x - M_y = 0,\ n=2)$
- We've worked examples of finding f when $\vec{\nabla} \times \vec{F} = \vec{0}$, via partial integration

We still need to prove the converse to the first bullet point, namely <u>Theorem B</u>, on page 5 of Wednesday notes.

(now it's page 2 of today's notes)

We may wish to discuss page 6 first, however, which gives a physics reason why gradient vector fields are so important.

It's also important to know

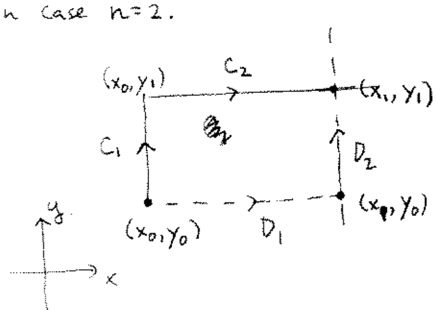
__Theorem B__ If, for the continuous vector field $\vec{F}$, line integrals only depend on initial and final points of the path, then the line integral itself can be used to find a scalar function f with $\nabla f = \vec{F}$.

In fact, __define__ $f(\vec{x}) = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ where $\mathcal{C}$ is __any__ path connecting the variable $\vec{x}$ to a fixed base point $\vec{x}_0$

(ℭ is arbitrary since our hypothesis is that line integrals are path independent.)

then $\nabla f = \vec{F}$.

proof: in case $n=2$.



$\vec{F} = \langle M, N \rangle$

thus

$$f(x_1,y_1) = \int_{C_1} N\,dy + \int_{x_0}^{x} M(t,y_1)\,dt$$

so $\frac{\partial f}{\partial x}(x_1,y_1) = 0 + M(x_1,y_1)$ (1210 FTC)

Also, $f(x_1,y) = \int_{D_1} M\,dx + \int_{y_0}^{y} N(x_1,t)\,dt$

so $\frac{\partial f}{\partial y}(x_1,y_1) = N(x_1,y_1)$ ∎

While the diagram for Theorem B is in our minds, let's prove

Theorem C For the configuration in Theorem B

$$\int_{\mathcal{C}_1 \cup \mathcal{C}_2} \vec{F}\cdot d\vec{r} = \int_{\mathcal{D}_1 \cup \mathcal{D}_2} \vec{F}\cdot d\vec{r}$$

holds if $\text{curl}(\vec{F}) = N_x - M_y = 0$ inside the rectangle.

Therefore, if $\text{curl}(\vec{F}) = 0$ in a domain where each point (x_1, y_1) and fixed basepoint (x_0, y_0) have such a rectangle still in the domain, then using rectangular paths we may deduce $\vec{F} = \nabla f$

proof

$$\left.\begin{aligned} \int_{\mathcal{C}_1} M\,dx + N\,dy &= \int_{y_0}^{y_1} N(x_0,y)\,dy \\ \int_{D_2} M\,dx + N\,dy &= \int_{y_0}^{y_1} N(x_1,y)\,dy \end{aligned}\right\}$$

$$\int_{D_2} \vec{F}\cdot d\vec{r} - \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} = \int_{y_0}^{y_1} N(x_1,y) - N(x_0,y)\,dy$$

$$\boxed{= \int_{y_0}^{y_1} \left(\int_{x_0}^{x_1} \frac{\partial N}{\partial x}(x,y)\,dx \right) dy.}$$

$$\left.\begin{aligned} \int_{\mathcal{C}_2} M\,dx + N\,dy &= \int_{x_0}^{x_1} M\,dx \quad (= M(x,y_1)) \\ \int_{D_1} M\,dx + N\,dy &= \int_{x_0}^{x_1} M(x,y_0)\,dx \end{aligned}\right\}$$

$$\int_{\mathcal{C}_2} \text{''} - \int_{D_1} \text{''} = \int_{x_0}^{x_1} \left(M(x,y_1) - M(x,y_0) \right) dx$$

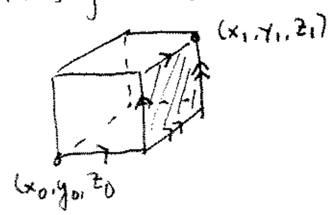
$$\boxed{= \int_{x_0}^{x_1}\int_{y_0}^{y_1} \frac{\partial M}{\partial y}(x,y)\,dy\,dx}$$

therefore,

$$\int_{\mathcal{D}_1} \vec{F}\cdot d\vec{r} + \int_{\mathcal{D}_2} \vec{F}\cdot d\vec{r} - \int_{\mathcal{C}_2} \vec{F}\cdot d\vec{r} - \int_{\mathcal{C}_1} \vec{F}\cdot d\vec{r} = \iint_{\text{rectangle}} \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dA = 0 \quad \text{if } \text{curl}\,\vec{F} \equiv 0 \,!$$

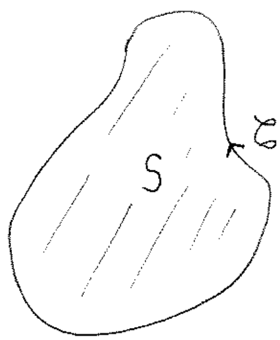
∎

This generalizes to $\mathbb{R}^3$: (face by face)



The argument on page 3 is a special case of <u>Green's Theorem</u> (§14.4), which we will prove the same way (on Monday).

<u>Green's Theorem</u> : Let $\vec{F} = \langle M, N \rangle$ be a continuously differentiable vector field defined in a region S. Let $\mathcal{C}$ be its boundary "∂S" (piecewise smooth), oriented counterclockwise (traverse $\mathcal{C}$ with S on your left.)



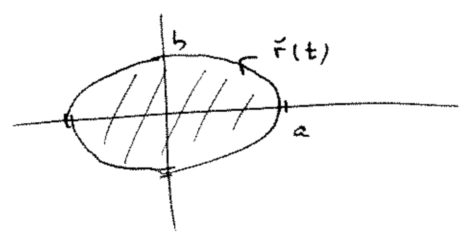
Then $\oint_{\mathcal{C}} M\,dx + N\,dy = \iint_S N_x - M_y \, dA$

$\uparrow$ scalar curl $\langle M, N \rangle$

<u>Example</u> : Verify the area formula for an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ using Green's Theorem.

Use $\vec{F}(t) = \langle a\cos t, b\sin t \rangle$
$0 \le t \le 2\pi$

and $\langle M, N \rangle = \langle -y, x \rangle$



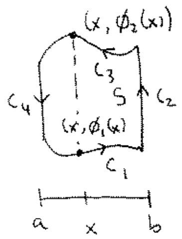
Proof of Green's Thm

① $\iint_S M_y \, dA = \oint_{\partial S} -M \, dx$

② $\iint_S N_x \, dA = \oint_{\partial S} N \, dy$

③ : ② − ① ⇒ $\iint_S N_x - M_y \, dA = \int_{\partial S} M\,dx + N\,dy$ ∎ (After ① ②).

① If S is y-simple:



$$\iint_S M_y \, dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} M_y \, dy \right) dx$$

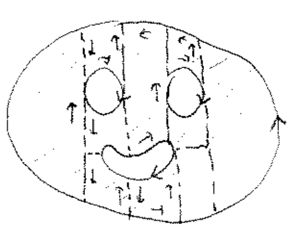
$$= \int_a^b M(x,y) \Big]_{\phi_1(x)}^{\phi_2(x)} dx$$

$$= \int_a^b M(x, \phi_2(x))\, dx - \int_a^b M(x, \phi_1(x))\, dx$$

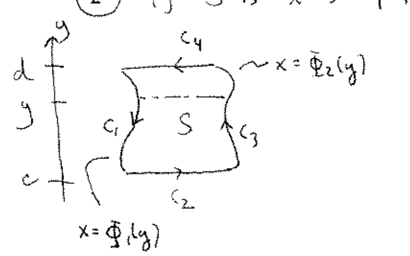
$$= -\int_{C_3} M\,dx - \int_{C_1} M\,dx$$

$$= -\oint_C M(x,y)\,dx$$ because $dx \equiv 0$ on C_4 & C_2!

For general S, decompose into vertically-simple subdomains and add up Green's identity over all of them. Line integrals on inside cancel out, giving result!



② If S is x-simple



$$\iint_S N_x \, dA = \int_c^d \int_{\Phi_1(y)}^{\Phi_2(y)} N_x(x,y)\,dx \; dy$$

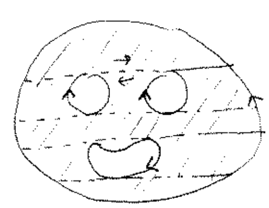
$$= \int_c^d N(x,y) \Big]_{x=\Phi_1(y)}^{x=\Phi_2(y)} dy$$

$$= \int_c^d N(\Phi_2(y), y)\,dy - \int_c^d N(\Phi_1(y), y)\,dy$$

$$= \int_{C_3} N(x,y)\,dy + \int_{C_1} N(x,y)\,dy$$

$$= \oint_C N(x,y)\,dy$$

General S:



∎

Physics application of gradient (aka conservative) vector fields

$\vec{F}$ = force field.

$\mathcal{C}$ connects A to B

$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x}$:= work done by $\vec{F}$ to move object from A to B along $\mathcal{C}$ (W)

PE := $-\int_C \vec{F}\cdot d\vec{x}$ = work done by object (Potential energy)

KE := $\frac{1}{2}mv^2$, $v = \|\vec{r}'(t)\|$, Kinetic energy.

If $\vec{F}$ is a gradient vector field ($\vec{F} = \nabla f$), it is called conservative, because if a particle moves by Newton ($\vec{F} = m\vec{r}''(t)$), then $\Rightarrow$ PE + KE $\equiv$ constant for partical motion.

proof

$$PE := -\int_{\vec{r}(0)}^{\vec{r}(t)} \vec{F}\cdot d\vec{x} = f(\vec{r}(0)) - f(\vec{r}(t))$$

$$KE \equiv \frac{1}{2}\, m\, \vec{r}'(t)\cdot\vec{r}'(t)$$

$$\frac{d}{dt}(KE+PE) = \frac{d}{dt}\left(\frac{1}{2}m\vec{r}'\cdot\vec{r}' + f(\vec{r}(0)) - f(\vec{r}(t))\right)$$

$$= m\vec{r}'\cdot\vec{r}'' - \nabla f(\vec{r}(t))\cdot\vec{r}'(t)$$

(product rule) (chain rule)

$$= \vec{r}'\cdot\Big[\underbrace{m\vec{r}'' - \vec{F}(\vec{r}(t))}_{=0 \text{ by Newton!}}\Big]$$

so total energy is conserved

examples

uniform grav. field

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix} = \nabla(-mgz)$$

PE = mgz (+ const)

inverse square law

$$\vec{F} = \left(-\frac{c}{|\vec{r}|^3}\right)\vec{r}$$

$$f = \frac{c}{|\vec{r}|}$$

PE = $-\frac{c}{|\vec{r}|}$ (+ const.)