Math 2210-3
Wed. Apr. 14

§ 14.3 Independence of path for line integrals

HW for Wed. April 21

14.3 1, 7, 9, 11, 17, 25, 26

In 17, do the computation both ways; via a path and with a potential function

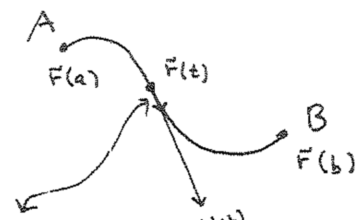
14.4 1, 6, 7, 9, 14, 16, 19

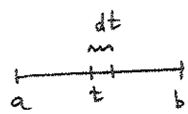
14.5 3, 7, 9, 10, 17, 19, 20

Recall

- If f is a scalar-valued function and C is a curve in space, then

$$\int_C f(\vec{x})\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$





$d\vec{x} = \vec{r}\,'(t)\,dt$

$ds = \|d\vec{x}\| = \|\vec{r}\,'(t)\|\,dt$

- If $\vec{F}$ is a vector field in space,

then $$\int_C \vec{F}(\vec{x})\cdot d\vec{x} = \int_a^b \left(\vec{F}(r(t))\cdot \vec{r}\,'(t)\right)dt$$

$$\left(= \int_a^b \left(\vec{F}(\vec{r}(t))\cdot \vec{T}\right)\|\vec{r}\,'(t)\|\,dt \quad \text{since the unit tangent vector } \vec{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \right.$$

$$\left. = \int_C (\vec{F}\cdot\vec{T})\,ds \quad \text{is a special case of the integral of a scalar function} \right)$$

- Notation: Our text writes

$\int_C \vec{F}(\vec{r})\cdot d\vec{r}$ for line integrals, whereas I wrote $\int_C \vec{F}(\vec{x})\cdot d\vec{x}$.

Both notations are common, since the curve $C = \{\vec{x} \text{ s.t. } \vec{x} = \vec{r}(t),\ a \le t \le b\}$.

The question of § 14.3 is:
For what vector fields $\vec{F}$ is it always true that

$\int_C \vec{F}\cdot d\vec{r}$ only depends on the initial point A, the terminal point B, and <u>not</u> on the choice of path C from A to B.

Answer: if and only $\vec{F}$ is a gradient vector field.
and in this case, the line integral's value is very easy to compute:
If $\vec{F} = \nabla f$, then $$\int_C \vec{F}(\vec{r})\cdot dr = f(B) - f(A) \quad !!$$

Example 1

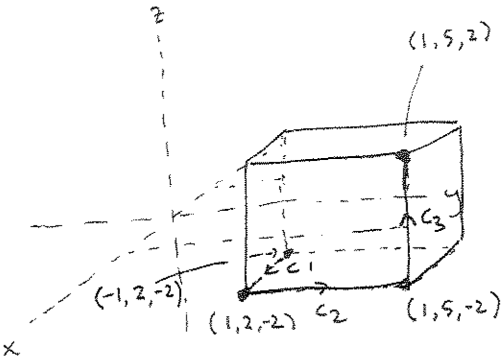
Let $\vec{F} = \langle yz^2,\ xz^2,\ 2xyz \rangle$

$A = (-1, 2, -2)$
$B = (1, 5, 2)$

1a) Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C yz^2\,dx + xz^2\,dy + 2xyz\,dz$

by following coord curves:

We should get 28



C_1: $-1 \le x \le 1$
$y = 2$
$z = -2$

C_2: $2 \le y \le 5$
$x = y$
$z = -2$

C_3: $-2 \le z \le 2$
$x = 1$
$y = 5$

Example 1 cont'd

1b) Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where C is a straight segment connecting $(-1,2,-2)$ to $(1,5,2)$

verify that this line integral also yields 28.

$$\vec{r}(t) = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad 0 \le t \le 1$$

$$\int_C \vec{F}\cdot d\vec{x} = \int_0^1 \left\langle \overset{y}{(2+3t)}\overset{z^2}{(-2+4t)^2},\ \overset{x}{(-1+2t)}\overset{z^2}{(-2+4t)^2},\ 2\overset{x}{(-1+2t)}\overset{y}{(2+3t)}\overset{z}{(-2+4t)} \right\rangle \cdot \overset{\vec{r}\,'(t)}{\langle 2, 3, 4 \rangle}\,dt$$

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> x:=-1+2*t:
  y:=2+3*t:
  z:=-2+4*t:
  int(2*y*z^2+3*x*z^2+4*2*x*y*z,t=0..1);
                                    28
>
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1c) $\vec{F} = \langle yz^2, xz^2, 2xyz \rangle$

notice that $\vec{F} = \nabla f$ for $f(x,y,z) = xyz^2$:

Thus $\displaystyle\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$

where $B = (1,5,2)$
$A = (-1,2,-2)$.

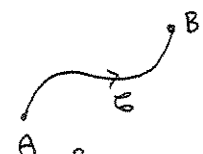
Do you get 28?

Is this way easier?

Notice, that for this vector field, you can now compute line integrals to your heart's content for whatever paths you're give, in an instant, as long as you you where the path begins and ends!

Why does this magic work?

<u>Theorem A</u> If $\vec{F} = \nabla f$, then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$



<u>proof</u>:

(so we sometimes write $\displaystyle\int_A^B \vec{F} \cdot d\vec{r}$ for the line integral)

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \quad \text{def}$$

$$= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt \quad \text{assuming } \vec{F} = \nabla f$$

$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \quad \text{chain rule chapter 12!}$$

$$= f(\vec{r}(t)) \Big]_{t=a}^{b} \quad \text{FTC!}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A) \quad \blacksquare$$

So how do you tell if a vector field is a gradient field?
We already saw, that if $\vec{F} = \nabla f$ then

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F} = \vec{0}$$

so this condition is necessary.
(In §14.4, 14.7 we see wh~~if~~ it's also essentially sufficient)

<u>Example 1 cont'd</u> $\vec{F} = \langle yz^2, xz^2, 2xyz \rangle$
1d) Verify $\vec{\nabla} \times \vec{F} = \langle 0,0,0 \rangle$.

Then use antidifferentiation or
definite integration to find (well, refind) f, so that $\nabla f = \vec{F}$.

It's also important to know

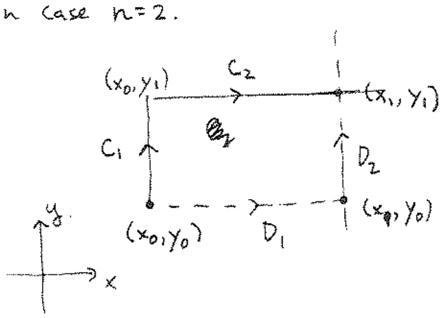
<u>Theorem B</u> If, for the continuous vector field $\vec{F}$, line integrals only depend on initial and final points of the path, then the line integral itself can be used to find a scalar function f with $\nabla f = \vec{F}$.

In fact, <u>define</u> $f(\vec{x}) = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ where $\mathcal{C}$ is <u>any</u> path connecting the variable $\vec{x}$ to a fixed base point $\vec{x}_0$

($\mathcal{C}$ is arbitrary since our hypothesis is that line integrals are path independent.)

then $\nabla f = \vec{F}$.

proof: in case $n=2$.



$\vec{F} = \langle M, N \rangle$

thus

$$f(x, y_1) = \int_{C_1} N\,dy + \int_{x_0}^{x} M(t, y_1)\,dt$$

so $\frac{\partial f}{\partial x}(x_1, y_1) = 0 + M(x_1, y_1)$ $\left(\begin{matrix}1210\\ \text{FTC}\end{matrix}\right)$

Also, $f(x_1, y) = \int_{D_1} M\,dx + \int_{y_0}^{y} N(x_1, t)\,dt$

so $\frac{\partial f}{\partial y}(x_1, y_1) = N(x_1, y_1)$ ∎