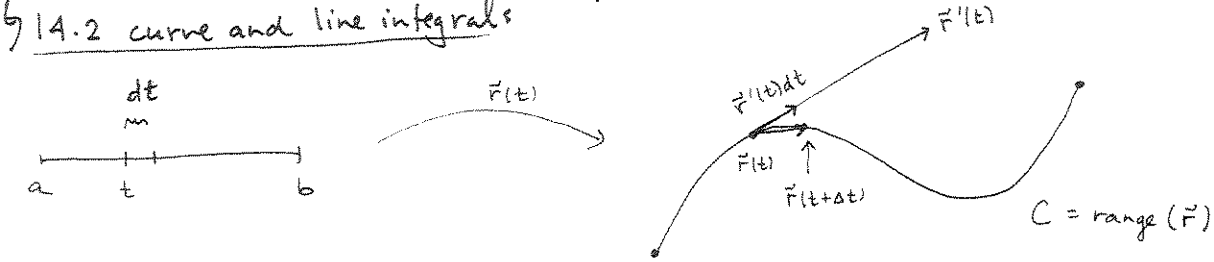
Math 2210-3
Monday April 12 ~ quickly go over divergence and curl computations from Friday notes, then...

§14.2 curve and line integrals



Recall from chapter 11:

$$\Delta\vec{r} = \vec{r}(t+dt) - \vec{r}(t) \approx d\vec{r} = \vec{r}\,'(t)\,dt$$

$$\Delta s = \|\vec{r}(t+dt) - \vec{r}(t)\| \approx ds = \|d\vec{r}\| = \|\vec{r}\,'(t)\|dt$$

ds is called the "element of arclength"

you computed curve length:

$$L = \int ds = \int_a^b |\vec{r}\,'(t)|\,dt$$

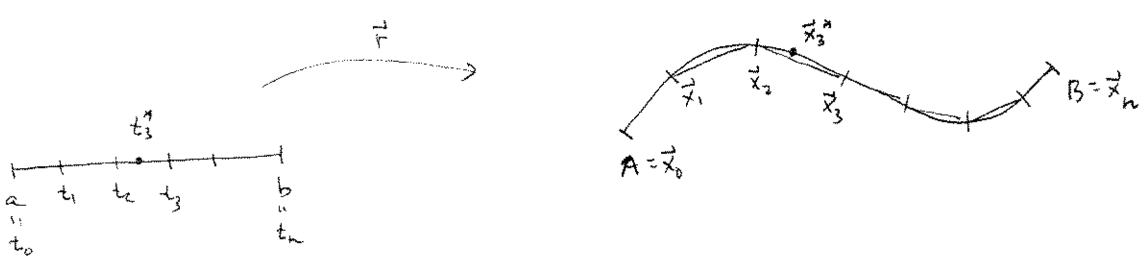
Curve integrals (of scalar functions)

if C is a curve in $\mathbb{R}^n$, the range of a parametric curve $\vec{r}(t)$ and f is a scalar function from C to $\mathbb{R}$,

$$\int_C f(x,y,z)\,ds := \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}\,'(t)|\,dt}_{ds}$$

Note: This integral is really a property of C, and not of the particular $\vec{r}(t)$ parameterizing it; this is because

$$\int_C f(\vec{x})\,ds = \lim_{\|P\|\to 0} \sum_i f(x_i^*)\,\underbrace{\|\vec{x}_i - \vec{x}_{i-1}\|}_{\Delta s_i}$$



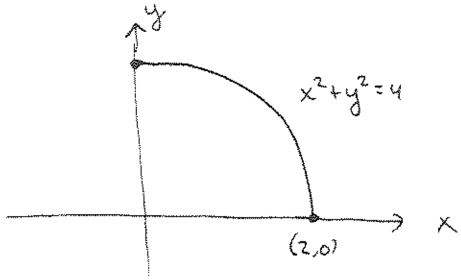
If $\vec{x}_i = \vec{r}(t_i)$

then the Riemann sum above is $= \sum_i f(\vec{r}(t_i^*))\,\|\vec{r}(t_{i})-\vec{r}(t_{i-1})\|$

$\approx \|\vec{r}\,'(t_{i-1})\|\,\Delta t_i$

fake application (but one way to visualize the integral) is as the (signed) area of a curtain above the curve C, if the curve is in x-y plane

more real
Example Let C be the radius 2 quarter-circle $x^2+y^2=4$
$x \geq 0, y \geq 0$



Suppose there is a wire in the shape of C, of density $\frac{xy}{2}$ (e.g. grams/meter)
mass/length.

Find mass $m = \int_C \underbrace{\delta \, ds}_{dm}$: Step 1: Find a good way to parameterize C!

center of mass $(\bar{x}, \bar{y})$ $\bar{x} = \frac{1}{m} \int_C x \, \delta \, ds$

$\bar{y} = \frac{1}{m} \int_C y \, \delta \, ds$

moments of intertia I_x, I_y, I_z

$I_x = \int_C y^2 \, \delta \, ds$

$I_y =$

$I_z =$

```
> mass:=int(4*cos(t)*sin(t),t=0..Pi/2);
  xbar:=int(8*cos(t)^2*sin(t),t=0..Pi/2)/mass;
  ybar:=int(8*cos(t)*sin(t)^2,t=0..Pi/2)/mass;
  Ix:=int(4*sin(t)^2*2*cos(t)*2*sin(t),t=0..Pi/2);
  Iy:=int(4*cos(t)^2*2*cos(t)*2*sin(t),t=0..Pi/2);
  Iz:=Ix+Iy;
```

$$mass := 2$$

$$xbar := \frac{4}{3}$$

$$ybar := \frac{4}{3}$$

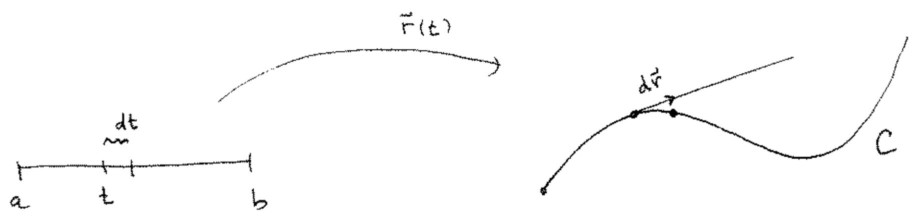
$$Ix := 4$$

$$Iy := 4$$

$$Iz := 8$$

```
>
```

<u>Line integrals</u> : A special type of curve integral use to compute work done by a force vector field. $\vec{F}(x,y,z)$

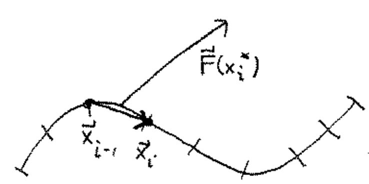


$$\Delta\vec{r} \approx d\vec{r} = \vec{r}\,'(t)\,dt$$

$$\int_C \vec{F} \cdot d\vec{r} \ := \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

← Really a property of the curve, and the direction you traverse it, since it's a limit of Riemann sums

$$\sum \vec{F}(\vec{x}_i^*) \cdot (\vec{x}_i - \vec{x}_{i-1})$$



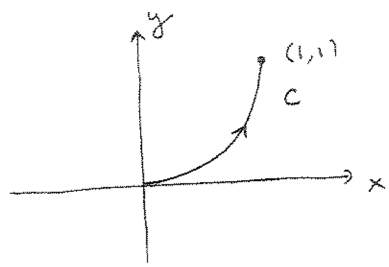
If we write
$\vec{F} = \langle M, N, P \rangle$
$d\vec{r} = \langle dx, dy, dz \rangle$
Then this integrand is also seen as

$$\int_C M\,dx + N\,dy + P\,dz \ := \int_a^b \left\{ M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + P(\vec{r}(t))\,z'(t) \right\} dt$$

$dx = x'(t)\,dt$
$dy = y'(t)\,dt$
$dz = z'(t)\,dt$

Example

Let C be the parabola $y=x^2$, going from $(0,0)$ to $(1,1)$
$0 \le x \le 1$



Let $\vec{F}(x,y) = \langle 2x+y, -x \rangle$

Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C (2x+y)\,dx - x\,dy$

Method 1 $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$

Method 2 $\vec{r}(x) = \langle x, x^2 \rangle$ $0 \le x \le 1$

(really method 1 all over, but explains dual notation for line ints)

Method 3 $\vec{r}(y) = \langle \sqrt{y}, y \rangle$