(1)

Name..

I.D. number ....

Math 2210-4

Practice Exam 2 Solutions

March 31, 2005

The exam will be closed-book and closed-note, but I will provide you with an integral table and any formulas which we have specifically agreed you won't need to memorize (e.g. mass and moment formulas). You will be allowed the use of a scientific calculator (only); you may not use a graphing calculator which can compute derivatives and integrals. (You won't actually need any calculator, but there could be times when numerical values would be a hint about whether you've correctly solved a problem.) **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. On the actual test there will be space for your work. **Good Luck!**

The problems below are only sample problems; they are **NOT** inclusive of all possible exam question topics.

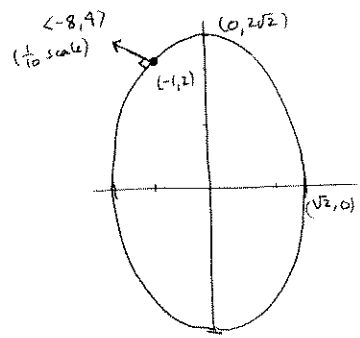
1) Let $f(x, y) = 4x^2 + y^2$.

1a) Sketch the level curve $f(x, y) = 8$. (5 points)

$4x^2+y^2=8$

$\frac{x^2}{2}+\frac{y^2}{8}=1$

ellipse



1b) Compute the gradient of $f(x, y)$, at the point (-1,2). Plot this gradient vector onto your level curve picture from (1a), using a length scale which is one tenth that of the x-y scale (so that your vector fits nicely onto your picture). How is the gradient vector related to the level curve at that point? (5 points)

$\nabla f = \langle f_x, f_y\rangle = \langle 8x, 2y\rangle$

$\nabla f(-1,2) = \langle -8, 4\rangle$

gradient of a function at a point is perpendicular to the level curve (or level surface) through it.

(2)

1c) In what unit direction is f decreasing most rapidly, at the point (-1,2)? (5 points)

$D_{\vec u} f(\vec p) = \nabla f(p)\cdot\vec u = |\nabla f(p)|\cos\theta$

is smallest when $\cos\theta=-1$, i.e. $\theta=\pi$, i.e. $\vec u$ is in direction opposite to $\nabla f(p)$. In this case $\boxed{\vec u = \langle 2,-1\rangle\frac{1}{\sqrt5}}$

1d) Compute the directional derivative $D_u(f)(-1, 2)$, where **u** is the unit vector in the same direction as <3,4>. (5 points)

$D_{\vec u} f(p) = \nabla f(p)\cdot\vec u$

$= \langle -8,4\rangle \cdot \frac{1}{5}\langle 3,4\rangle$

$= \frac{-24+16}{5} = \boxed{-\frac{8}{5}} = -1.6$

(1e) is 10 times (1d) because dir. the same, but speed is 10

1e) Let **r**(t) be a parametric curve passing through the point (-1,2) at time t=0, with speed 10, and with velocity vector in the same direction as <3,4>. Use the chain rule to compute the derivative of the composition f(**r**(t)), at t=0. How is your answer related to the answer to (1d), and why? (5 points)

$\frac{d}{dt} f(\vec r(t)) = \nabla f(\vec r(t))\cdot \vec r\,'(t)$

$= \langle -8,4\rangle \cdot 2\langle 3,4\rangle$

$= 2(-24+16) = -16$

$\vec r\,'(0) = 10\cdot\langle\frac35,\frac45\rangle = \langle 6,8\rangle$

↑ speed ↑ unit vector

1f) What is the equation for the tangent plane to the graph z=f(x,y), at the point (-1,2,8)? (5 points)

$z = f_x(p)(x-x_0) + f_y(p)(y-y_0) + z_0$

$\nabla f(-1,2) = \langle -8,4\rangle$

$\boxed{z = 8 - 8(x+1) + 4(y-2)}$

1g) Use differential (or tangent function) approximation to estimate f(-.9,1.8), using f(-1,2)=8. (5 points)

$x=-1 \quad dx = .1$

$y=2 \quad dy = -.2$

$dz = f_x\,dx + f_y\,dy$

$= (-8)(.1) + 4(-.2)$

$= -1.6$

$f(x+dx, y+dy) \approx f(x,y) + dz$

$= 8 - 1.6$

$\boxed{= 6.4}$

(exact value $f(-.9,1.8) = 6.48$)

(3)

2) Use Lagrange multipliers to find the maximum value of $x^2 - y^2$, subject to the constraint that $x^2 + y^2 = 4$.

(20 points)

$\nabla f = \lambda \nabla g$

$\langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$

$\left.\begin{matrix} 2x = \lambda 2x \\ -2y = \lambda 2y \end{matrix}\right\}$

$x = 0 \Rightarrow y = \pm 2 \Rightarrow f(0, \pm 2) = -4$

$x \neq 0 \Rightarrow \lambda = 1 \Rightarrow y = -y \Rightarrow y = 0 \Rightarrow x = \pm 2 \Rightarrow f(\pm 2, 0) = 4$ max value.

[If you'd done this in calc, you'd say $y^2 = 4 - x^2$, so $x^2 - y^2 = x^2 - (4 - x^2) = 2x^2 - 4$ is clearly max when x^2 is max]

3a) Compute

$$\int_0^2 \int_{x^2}^4 x y^2 \, dy \, dx$$

(10 points)

$\left.\frac{xy^3}{3}\right]_{y=x^2}^{4}$

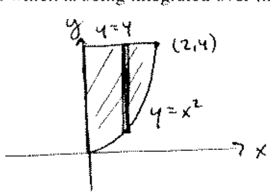
$\int_0^2 \frac{x}{3} \cdot 64 - \frac{x}{3} x^6 \, dx = \left.\frac{32}{3}x^2 - \frac{x^8}{24}\right]_0^2 = \frac{32 \cdot 4}{3} - \frac{2^8}{24} = \frac{128}{3} - \frac{2^5}{3} = \frac{96}{3} = 32$

3b) Sketch the region in the plane which is being integrated over in 3a).

(5 points)

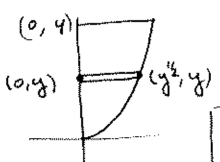
$0 \le x \le 2$

$x^2 \le y \le 4$



3c) Express the double integral in (3a) as an iterated integral with the order of integration reversed. You need not recompute its value, although if you have time it could be a useful check.

(10 points)



$y = x^2$

$x = \sqrt{y}$ (if $x > 0$)

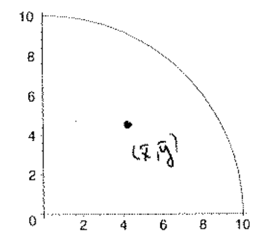
$$\int_0^4 \int_0^{\sqrt{y}} x y^2 \, dx \, dy$$

(4)

4) A laminate in the shape of a quarter disk of radius 10 cm has density function

$$\delta(x, y) = .1 (x^2 + y^2)^{\left(\frac{1}{2}\right)} \frac{gm}{cm^2}.$$

Here is a picture of the laminate (not to scale):



Compute the mass and the center of mass of this laminate. (Note: since the laminate is symmetric with respect to the line y=x, you may use the fact that the center of mass will lie on this line.) Hint: the computations will work out much more nicely in polar coordinates.

(20 points)

$m = \iint_D \delta \, dA = \int_0^{\pi/2} \int_0^{10} \underbrace{\frac{r}{10}}_{\delta} \underbrace{r \, dr \, d\theta}_{dA} = \int_0^{\pi/2} \left.\frac{1}{10} \frac{1}{3} r^3 \right]_0^{10} d\theta = \frac{100}{3} \cdot \frac{\pi}{2} = \boxed{\frac{50\pi}{3} \text{ gm}}$

$\bar{x} = \bar{y}$ (see above)

$M_y = \iint_D x \, \delta \, dA = \int_0^{\pi/2} \int_0^{10} r\cos\theta \, \frac{r}{10} \, r \, dr \, d\theta = \int_0^{\pi/2} \left.\frac{1}{10} \cos\theta \frac{r^4}{4}\right]_0^{10} d\theta = \int_0^{\pi/2} 250 \cos\theta \, d\theta = \left. 250 \sin\theta \right]_0^{\pi/2} = 250$

$\bar{x} = \frac{M_y}{m} = \frac{250}{\frac{50\pi}{3}} = \frac{5 \cdot 3}{\pi} = \frac{15}{\pi}$

$\boxed{\bar{x} = \bar{y} = \frac{15}{\pi}} \approx 4.77$ cm.