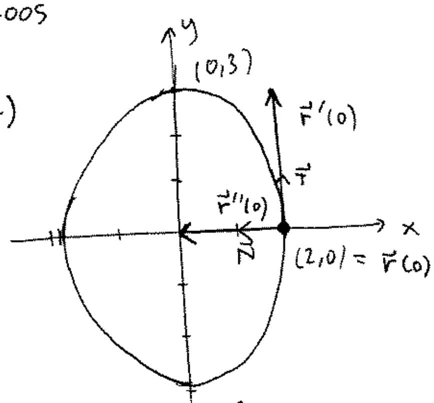
Math 2210-4
Practice Final Solutions
4/2005

1\. a)



$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

b) $\vec r(t) = \langle 2\cos t, 3\sin t\rangle$, with $x = 2\cos t$, $y = 3\sin t$

for points on this curve

$$\frac{x^2}{4}+\frac{y^2}{9} = \frac{\cancel{4}\cos^2 t}{\cancel{4}} + \frac{\cancel{9}\sin^2 t}{\cancel{9}} = 1$$

so each point with position vector $\vec r(t)$ lies on the ellipse

c) $\vec r(t) = \langle 2\cos t, 3\sin t\rangle$ $\qquad \vec r(0) = \langle 2, 0\rangle$

$\vec r\,'(t) = \langle -2\sin t, 3\cos t\rangle$ $\qquad \vec r\,'(0) = \langle 0, 3\rangle$

$\vec r\,''(t) = \langle -2\cos t, -3\sin t\rangle$ $\qquad \vec r\,''(0) = \langle -2, 0\rangle$

d) $\vec T = \dfrac{\vec r\,'}{|\vec r\,'|}$; $\vec T(0) = \langle 0, 1\rangle$

$\vec N$ points on side of curve to which curve is bending, $\perp \vec T$

so $\vec N(0) = \langle -1, 0\rangle$ ("flip & − trick")

e) clearly $\vec r\,''(0) = 0\vec T + 2\vec N$

exact values: $\vec r\,''(0)\cdot\vec T(0) = \langle -2,0\rangle\cdot\langle 0,1\rangle = 0$ = tangential component of acceleration

$\vec r\,''(0)\cdot\vec N = \langle -2,0\rangle\cdot\langle -1,0\rangle = 2$ = normal component of acceleration

f) $f(x,y) = \dfrac{x^2}{4}+\dfrac{y^2}{9}$; $\vec r(t) = \langle 2\cos t, 3\sin t\rangle$

$$\frac{d}{dt} f(\vec r(t)) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} \qquad (= \nabla f(\vec r(t))\cdot \vec r\,'(t))$$

$$= \frac{x}{2}(-2\sin t) + \frac{2}{9}y(3\cos t)$$

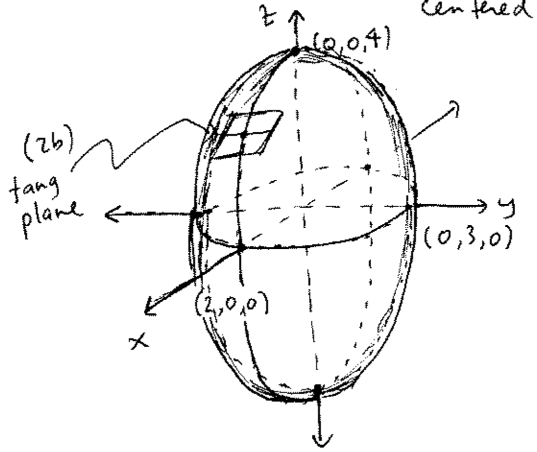
$$= \cos t(-2\sin t) + \frac{2}{9}(3\sin t)(3\cos t)$$

$$= -2\cos t\sin t + 2\sin t\cos t = 0.$$

Since $\vec r(t)$ lies on our ellipse we know $f(\vec r(t)) \equiv 1$

so $\dfrac{d}{dt} f(\vec r(t)) \equiv 0$!

2a) $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ is an ellipsoid; traces in each coord plane are ellipses centered at origin



2b) ellipsoid is a level surface $f(x,y,z)=1$

gradients are $\perp$ to level surfaces!

$\nabla f = \langle \frac{x}{2}, \frac{2y}{9}, \frac{z}{8} \rangle$

$\nabla f(\sqrt{3}, 0, 2) = \langle \frac{\sqrt{3}}{2}, 0, \frac{1}{4} \rangle \ /\!/ \ \langle 2\sqrt{3}, 0, 1 \rangle$

is normal vector

$2\sqrt{3}(x - \sqrt{3}) + 0(y-0) + (z-2) = 0$

or $2\sqrt{3}x + z = 8$

3. $\oint_C \vec{F}\cdot d\vec{r} = \iint_S N_x - M_y \, dA$ Green's theorem

$\vec{F} = \langle M, N \rangle = \langle -y, x \rangle$

so $\iint N_x - M_y \, dA = \iint 2 \, dA = \boxed{2 \text{ (area of ellipse)}}$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \langle -3\sin t, 2\cos t \rangle \cdot \langle -2\sin t, 3\cos t \rangle \, dt$

$\vec{r}(t) = \langle 2\cos t, 3\sin t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 3\cos t \rangle$

$= \int_0^{2\pi} 6(\underbrace{\sin^2 t + \cos^2 t}_{1})\,dt = 6\cdot 2\pi \ \boxed{= 12\pi}$

So $2A = 12\pi$; $\boxed{A = 6\pi}$

4 a) $\int_0^4 \int_{y/2}^{\sqrt{y}} 4x \, dx \, dy = \int_0^4 2x^2 \Big]_{y/2}^{\sqrt{y}} dy$

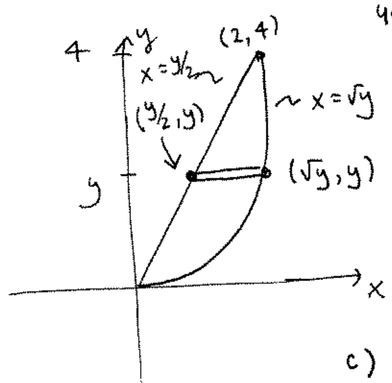
$= \int_0^4 2y - 2\left(\frac{y^2}{4}\right) dy = y^2 - \frac{y^3}{6} \Big]_0^4 = \underbrace{16}_{48/3} - \underbrace{\frac{64}{6}}_{32/3} = \boxed{16/3}$

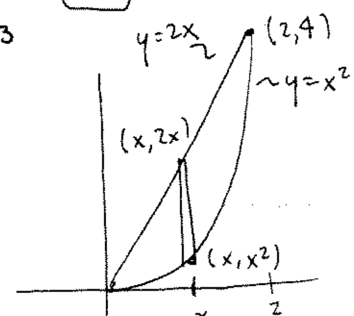
b) $0 \le y \le 4$

for fixed y,

$y/2 \le x \le \sqrt{y}$

$y = 2x \quad x^2 = y$





c) $\int_0^2 \int_{x^2}^{2x} 4x \, dy \, dx = \int_0^2 4x(2x - x^2)\,dx$

$= \int_0^2 8x^2 - 4x^3 dx = \frac{8}{3}x^3 - x^4 \Big]_0^2$

$= \frac{64}{3} - 16 = \boxed{\frac{16}{3}}$

5 a) (i) $\langle M, N\rangle = \langle \sin x + e^x\cos y,\ 3y^2 - e^x\sin y\rangle$

$M_y = -e^x\sin y = N_x$; this is a gradient field.

(ii) $\langle M, N\rangle = \langle -y, x\rangle$

$\left.\begin{array}{l} M_y = -1 \\ N_x = 1 \end{array}\right\} \neq$ NOT a gradient field

b) for (i), $f_x = \sin x + e^x\cos y$

$\Rightarrow f = \int f_x\,dx = -\cos x + e^x\cos y + C(y)$

$\Rightarrow f_y = -e^x\cancel{\sin y} + C'(y) = 3y^2 - e^x\cancel{\sin y} \Rightarrow C'(y) = 3y^2$

$\Rightarrow C(y) = y^3 + \text{const}$

$\Rightarrow \boxed{f(x,y) = -\cos x + e^x\cos y + y^3}$ $(+c)$

c) $\int_A^B \vec{F}\cdot d\vec{r} = f(B) - f(A)$

$= f(\pi/2, \pi/2) - f(0,0)$

$= \left(\frac{\pi}{2}\right)^3 - [-1+1+0] = \left(\frac{\pi}{2}\right)^3$

6. $f(x,y,z) = xy^2/1+z^2$ $\qquad D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z)\cdot\vec{u}$

So f increases most rapidly in dir of grad f

$\nabla f = \left\langle \frac{y^2}{1+z^2},\ \frac{2xy}{1+z^2},\ \frac{-2xy^2z}{(1+z^2)^2}\right\rangle$

$\nabla f(1,1,1) = \left\langle \frac{1}{2}, \frac{2}{2}, \frac{-2}{4}\right\rangle \parallel \langle 1,2,-2\rangle$

a) unit dir $\boxed{\vec{u} = \frac{1}{3}\langle 1,2,-2\rangle}$ $\qquad (|\langle 1,2,-2\rangle| = \sqrt{9} = 3)$

b) $f(1.01, 1.98, 2.03) \approx f(1,2,2) + f_x\,dx + f_y\,dy + f_z\,dz$

$\nabla f(1,2,2)$ (oops, I meant the point in (a), (b) to be the same!)

$\parallel$

$\left\langle \frac{4}{5}, \frac{4}{5}, \frac{-16}{25}\right\rangle = \langle .8, .8, -.64\rangle$

$f(1,2,2) = 4/5 = .8$ $\qquad dx = .01$

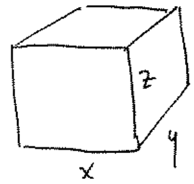
$dy = -.02$

So, $f(1.01, 1.98, 2.03)$ $\qquad dz = .03$

$\approx .8 + .01(.8) - .02(.8) + .03(-.64)$

$\boxed{= .7728}$

7.



$$xyz = 60$$

$$\text{cost} = \underset{\text{top}}{2xy} + \underset{\text{bottom}}{3xy} + 1\,(2yz+2xz) = 5xy + 2yz + 2xz$$

$z = \frac{60}{xy}$, minimize $f(x,y) = 5xy + 2y\frac{60}{xy} + 2x\frac{60}{xy}$

$$= 5xy + \frac{120}{x} + \frac{120}{y}$$

critical pts $\quad f_x = 0 = 5y - \frac{120}{x^2} \qquad \frac{120}{x^2} = 5y;\quad x^2y = \frac{120}{5} = 24$

$\qquad f_y = 0 = 5x - \frac{120}{y^2} \qquad \frac{120}{y^2} = 5x \quad xy^2 = \frac{120}{5} = 24$

$\frac{x^2y}{xy^2} = 1 \quad \boxed{x=y}$

$x^3 = 24$

$$\boxed{\begin{aligned} x &= 2\cdot 3^{1/3}\\ y &= 2\cdot 3^{1/3}\\ z &= \frac{60}{4\cdot 3^{2/3}} = 5\cdot 3^{1/3}\end{aligned}}$$

or, with Lagrange multipliers

$\nabla \text{cost} = \lambda \nabla \text{constraint}$

$\langle 5y+2z,\ 5x+2z,\ 2y+2x\rangle$

$= \lambda\langle yz,\ xz,\ xy\rangle$

$5y+2z = \lambda yz$
$5x+2z = \lambda xz$
$2y+2x = \lambda xy$

$\lambda = \frac{5y+2z}{yz} = \frac{5}{z} + \frac{2}{y}$

$\lambda = \frac{5x+2z}{xz} = \frac{5}{z} + \frac{2}{x}$

$\lambda = \frac{2y+2x}{xy} = \frac{2}{x} + \frac{2}{y}$

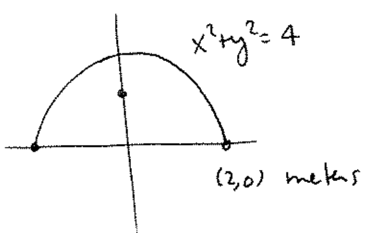
$\boxed{x=y}$ leads to same answer since

$\frac{5}{z} + \frac{2}{x} = \frac{4}{x}$

$\frac{5}{z} = \frac{2}{x};\quad \boxed{z = \frac{5}{2}x}$

$xyz = 60$
$\Rightarrow \frac{5}{2}x^3 = 60$
$\Rightarrow x^3 = 60\cdot\frac{2}{5}$
$= 24.$

8.



$\delta = 2$ gm/meter

mass = (length) (density) $\quad \cancel{m}\cdot \text{gm}/\cancel{m}$

$= \frac{1}{2}(2\pi\cdot 2)(2) \quad$ gm

(a) $\boxed{m = 4\pi \text{ gm.}}$

(b) $\boxed{\bar{x} = 0}$ by symmetry

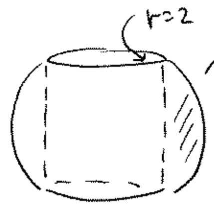
$$\bar{y} = \frac{\int_C y\delta\, ds}{\int_C \delta\, ds}$$

$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle;\quad 0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t\rangle$

$ds = |\vec{r}\,'(t)|\,dt = 2\,dt$

So, $\bar{y} = \frac{1}{4\pi}\int_0^\pi 2\cdot 2\sin t\cdot 2\,dt = \frac{2}{\pi}\int_0^\pi \sin t\,dt = \boxed{\frac{4}{\pi}}$ makes sense.

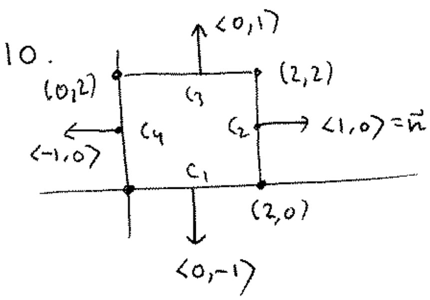
9. 

cylindrical coords!

$$2 \le r \le 3$$
$$0 \le \theta \le 2\pi$$

for fixed r,θ, $\quad -\sqrt{9-r^2} \le z \le \sqrt{9-r^2}$

$$\text{Vol} = \iiint 1 \, dV$$
$$= \int_0^{2\pi}\int_2^3\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} 1\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_2^3 2r(9-r^2)^{1/2}\,dr\,d\theta$$
$$-\frac{2}{3}(9-r^2)^{3/2}\Big]_2^3$$
$$= \int_0^{2\pi} \frac{2}{3}\cdot 5^{3/2}\,d\theta = \boxed{\frac{4\pi}{3}5^{3/2}}$$

10. 

$$\vec{F} = \langle x,y\rangle$$

a) $$\int_{\square} \vec{F}\cdot\vec{n}\,ds = \int_{C_1}\langle x,0\rangle\cdot\langle 0,-1\rangle\,ds \quad = 0$$
$$+ \int_{C_2}\overbrace{\langle 2,y\rangle\cdot\langle 1,0\rangle}^{2}\,ds \quad + 4 \qquad \boxed{=8}$$
$$+ \int_{C_3}\overbrace{\langle x,2\rangle\cdot\langle 0,1\rangle}^{2}\,ds \quad + 4$$
$$+ \int_{C_4}\overbrace{\langle 0,y\rangle\cdot\langle -1,0\rangle}^{0}\,ds \quad + 0$$

b) $$\int_{\square}\vec{F}\cdot\vec{n}\,ds = \iint_{\square} \operatorname{div}\vec{F}\,dA$$

$\operatorname{div}\vec{F} = 2$, so

$$\iint_{\square} 2\,dA = 2\,(\text{area})$$
$$= 2\cdot 4 = \boxed{8}$$