Math 2210-4
Monday 3/7

Review!

$f$ diff'ble at $\vec{p}_0$ :

$$f(\vec{p}_0+\vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(h) \qquad \text{where } \vec{\varepsilon}(\vec{h})\to 0 \text{ as } \vec{h}\to 0$$

led to

$$D_{\vec{u}}f(\vec{p}_0) := \lim_{h\to 0}\frac{f(p_0+h\vec{u})-f(p_0)}{h} \qquad (|\vec{u}|=1) \qquad \text{Directional derivative.}$$

$$= \nabla f(\vec{p}_0)\cdot\vec{u}$$

$$\frac{d}{dt}f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot\vec{r}\,'(t) \qquad \text{Chain rule}$$

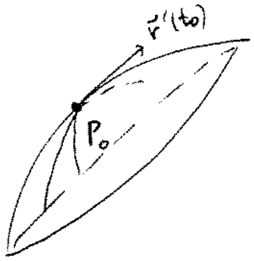
- Notice how similar the chain rule formula is to the directional derivative formula, and that this is sensible.

<u>Example</u> : One fine spring day is SLC,
the temperature on the East Bench is approximately

$$T(x,y) = 50 - x - y - .2xy - .5x^2 - \tfrac{1}{3}y^3 \qquad \text{degrees F}$$

where $(x,y)$ are EW/NS coords in miles from downtown.
At this instant a student is at $(2,0)$ and ~~walking~~ biking northeast at 4 mph. At what rate is the temperature of the student's surroundings changing?

**Theorem**: Let $F(x,y,z)=k$ be the equation of a level surface in $\mathbb{R}^3$, for the function $F(x,y,z)$.
Let $(x_0,y_0,z_0)$ be on this level surface. Then $\nabla f(x_0,y_0,z_0)$ is a normal vector to the tangent plane to the surface at $(x_0,y_0,z_0)$




$F(x,y,z)=k$
(piece of surface

proof. Let $\vec{r}(t)$ be a curve lying on the level surface, passing thru $\vec{p}_0=(x_0,y_0,z_0)$ at time $t_0$

Then $F(\vec{r}(t)) \equiv k$

so $\frac{d}{dt} F(\vec{r}(t)) \equiv 0$

$\|$ chain rule

$\nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

at $t_0$, $\nabla f(\vec{p}_0) \cdot \vec{r}'(t_0)=0$

$\Rightarrow \nabla f(\vec{p}_0) \perp \vec{r}'(t_0)$.

But this is true for <u>all</u> possible curves $\vec{r}(t)$ on M, passing thru p,

so $\nabla f(\vec{p}_0) \perp$ to <u>all</u> tangents vectors to M at p.

(Notice the idea of this proof also shows that for a level curve $f(x,y)=k$ at $(x_0,y_0)$ on this curve, $\nabla f(x_0,y_0)$ is $\perp$ to the curve.)

**Thus**: The equation of the tangent plane at $(x_0,y_0,z_0)$ is

$$\boxed{\begin{array}{l} \nabla F(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0 \\ \text{i.e.} \quad F_x(\vec{p}_0)(x-x_0) + F_y(\vec{p}_0)(y-y_0) + F_z(\vec{p}_0)(z-z_0) = 0 \end{array}}$$

<u>Example</u>: Find the eqtn for the tangent plane to

$$\frac{x^2}{4} + \frac{y^2}{8} + \frac{z^2}{16} = 1$$

at $(1, 2, 2)$

(a): Solve for $z(x,y)$, use old formula. (Hard!)

(b) Use old formula, but find $z_x$, $z_y$ implicitly. (better)

(c) Use Result from last page (best).

## Differential approximation

Using the tangent approximation $T(x)$ to $f(x)$ near $x_0$, $T(x,y)$ to $f(x,y)$ near $(x_0,y_0)$, $T(x,y,z)$ to $f(x,y,z)$ near $(x_0,y_0,z_0)$

and we did some examples of these last Monday §15.4.

is also called differential approximation, and is often written using "differentials."

1210 : $y = f(x)$ You've seen this!

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \Delta x\, \varepsilon(x)$$

$dx := \Delta x$

$\Delta y := f(x+\Delta x) - f(x)$

$\boxed{dy := f'(x)\,dx}$ is the tangent function approx to $\Delta y$

2210 $z = f(x,y)$

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y + \Delta x\, \varepsilon_1(\Delta x,\Delta y) + \Delta y\, \varepsilon_2$$

$dx := \Delta x$

$dy := \Delta y$

$\Delta z = \Delta f := f(x+\Delta x, y+\Delta y) - f(x,y)$

$\boxed{dz := f_x\,dx + f_y\,dy}$ is the tangent approx to $\Delta z$

etc.

**Remark**: differentials are compatible with the chain rule. Notice that in both 1210 & 2210 if $x$ $(x,y)$ are fns of $t$, get chain rule by dividing either box by $dt$

**Example**: A steel cylinder is measured to be 10 cm tall, ($\pm .1$ cm) with radius 2 cm ($\pm .2$ cm).

- Estimate the volume, with error bounds derived from differentials.
- Compare with exact error bound.

48.884
32.076