Math 2210-4
Friday 4 March

§15.6 Chain Rule

Homework for Friday 3/10

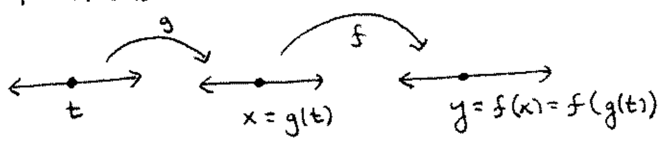
15.6 1, 6, 12, 13, 17, 18, 19, 20, 29
15.7 1, 4, 6, 9, 15, 22
15.8 1, 3, 7, 11, 12, 13, 15, 16, 19, 20, 22, 23, 28, 29

correct eqtns are

$$m \sum x_i^2 + b \sum x_i = \sum x_i y_i$$
$$m \sum x_i + nb = \sum y_i$$

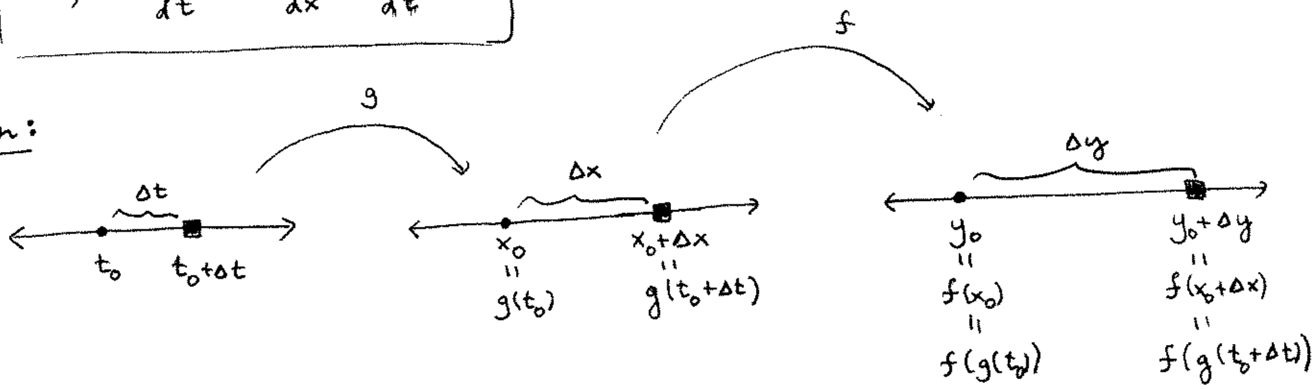
my book has $\sum x_i y_i$ here, which is wrong.

1210: functions of 1-variable



$$\boxed{\begin{aligned} &\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t) \\ &\text{or,} \quad \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \end{aligned}}$$

Reason:



Tangent approximation formula for f:

$$\Delta y = f'(x_0)\Delta x + (\Delta x)\,\varepsilon(\Delta x)$$

$$\frac{\Delta y}{\Delta t} = f'(x_0)\frac{\Delta x}{\Delta t} + \left(\frac{\Delta x}{\Delta t}\right)\varepsilon(\Delta x)$$

take $\lim\limits_{\Delta t \to 0}$: $\quad \dfrac{dy}{dt} = f'(g(t_0))\, g'(t_0) + 0$

$\frac{\Delta x}{\Delta t}\,\varepsilon(\Delta x) \to g'(t_0)\cdot 0$

since $\Delta x \to 0$ as $\Delta t \to 0$.

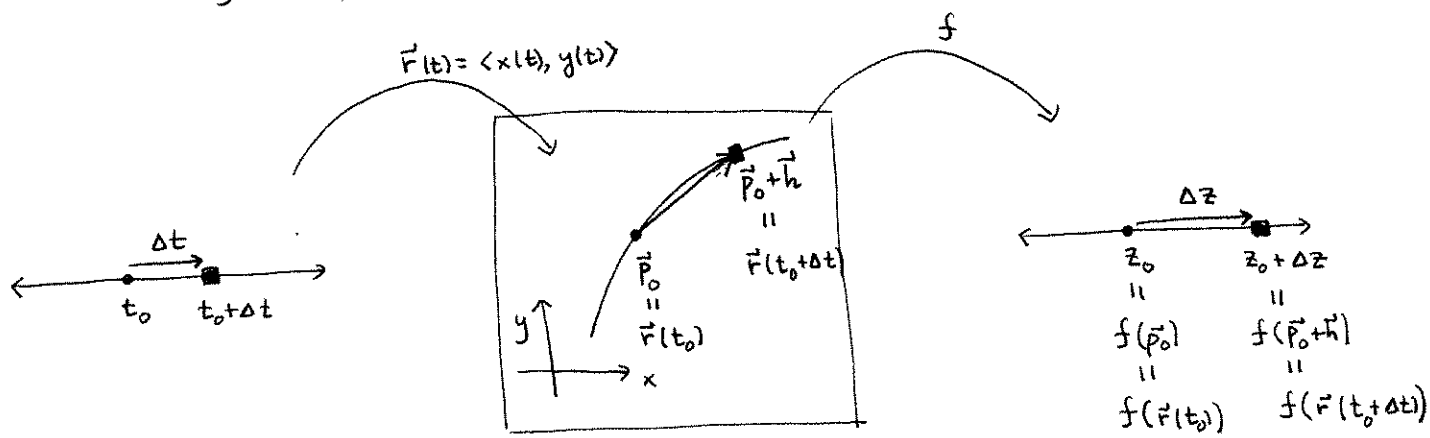
multivariable:

$$\mathbb{R} \xrightarrow{\vec{r}(t)} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

t	(x,y)	z
	(x,y,z)	w
	etc	

$$\boxed{\begin{array}{l} \textbf{Chain rule} \\ \dfrac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t) \\ \text{i.e.} \quad \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} \\ \qquad \dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} \\ \qquad \text{etc.} \end{array}}$$

Reason is tangent approximation formula!



affine approx for f

$$f(\vec{p}_0 + \vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{h} \cdot \vec{\varepsilon}(\vec{h})$$

$$\Delta z = f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0) = \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{h} \cdot \vec{\varepsilon}(\vec{h})$$

$$\frac{\Delta z}{\Delta t} = \nabla f(\vec{p}_0) \cdot \frac{\vec{h}}{\Delta t} + \frac{\vec{h}}{\Delta t} \cdot \vec{\varepsilon}(\vec{h})$$

Note $\dfrac{\vec{h}}{\Delta t} = \dfrac{\vec{r}(t_0 + \Delta t) - \vec{r}(t_0)}{\Delta t}$

$\longrightarrow \vec{r}\,'(t_0)$
as $\Delta t \to 0$

Let $\Delta t \to 0$:

$$(f \circ r)'(t_0) = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0) \qquad + \vec{r}\,'(t_0) \cdot \vec{0}$$

because $\vec{h} \to 0$ as $\Delta t \to 0$
so $\varepsilon(\vec{h}) \to 0$ too.

∎

Examples

$z = x^3 y \qquad x = 2t$
$\qquad\qquad\qquad\ \ y = t^2$

$$\begin{tikzcd} & z & \\ x \arrow[ur] & & y \arrow[ul] \\ & t \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Find $\frac{dz}{dt}$

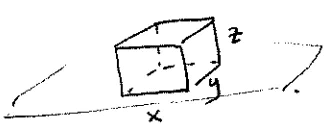
(a) by page 2 chain rule

(b) by writing z as a fcn of t

Example (Related rates revisited)

A block of ice is melting in the sun, remaining in the shape of a rectangular prism.

At noon the height z is $12''$ and is decreasing at a rate of $1''$/hour.



depth y is $20''$, decreasing @ $\frac{1}{2}''$/hour
length x is $30''$, decreasing @ $\frac{1}{2}''$/hour.

How fast is volume changing?

$V = xyz$

chain rule for partial derivatives is identical : (really this is an application of what we've already done.)

i.e.
e.g.

$$\begin{pmatrix} r \\ s \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to w$$

w is a fcn of (x,y,z)
x,y,z are each fcns of r,s.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

<u>Example</u>

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$z = x^2 + y^2$$

Find z_r
z_θ

with the chain rule
and by direct substitution.

It takes practice to make the chain rule feel natural!