Math 2210
Monday 3/21

Integration!
§16.1-16.2
Double integrals

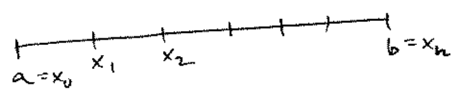
HW for Fri 3/25

assigned before break
- 15.8 28, 29 (book typo on 28)
- 15.9 1, 3, 5, 9, 10, 11 (#9 both ways, §15.8, §15.9)

new!
- 16.1 1, 5, 8, 9, 10, 16
- 16.2 5, 7, 11, 12, 13, 14, 17, 18

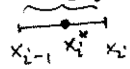
$\boxed{\text{Remember 1210 definite integral}}$

$$\int_a^b f(x) := \lim_{|P|\to 0} \underbrace{\sum_{i=1}^{n} f(x_i)\,\Delta x_i}_{\text{Riemann sum}}$$



i^{th} subinterval

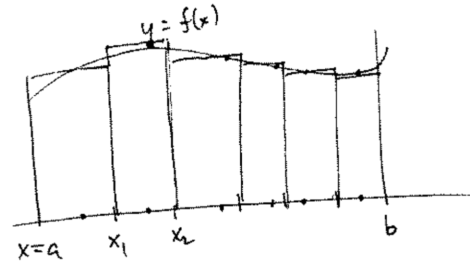
$\underbrace{}\ \Delta x_i = x_i - x_{i-1}$



$|P| = \max(\Delta x_i)$
↑
"norm of partition"

we motivated this def'n by thinking about ("signed") area under a graph $y=f(x)$, but the definite integral had lots of other applications:

- area between graphs
- work, if $f(x)$ = force
- curve length
- volumes (e.g. by slicing, e.g. for surfaces of revolution)
- surface areas of revolution
- centers of mass



- If f is continuous on $[a,b]$ then the definite integral exists
- Fundamental Thm of Calculus says you can compute

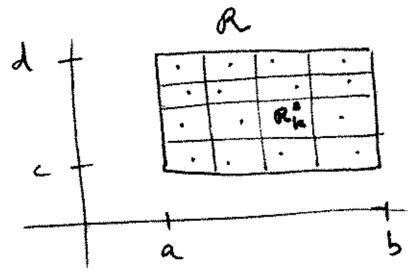
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antideriv. of f.

Double integrals over rectangles

Let $R = \{(x,y): a \le x \le b,\ c \le y \le d\}$
be a coordinate rectangle.

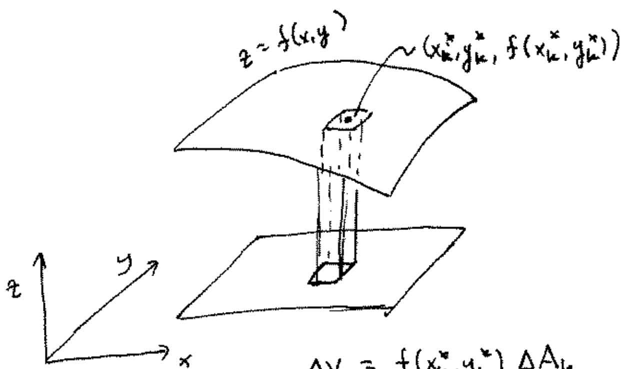
$f(x,y)$ defined for $(x,y) \in R$.



Let P be a partition of R into subrectangles R_k, of area ΔA_k with sample points (x_k^*, y_k^*) in R_k

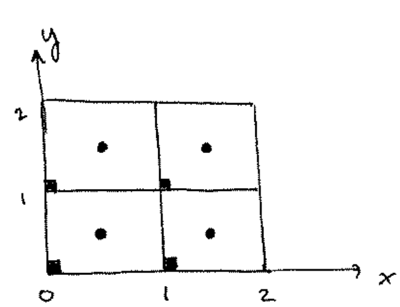
Then $\sum_k f(x_k^*, y_k^*)\,\Delta A_k$ is the Riemann sum for this partition P, for f.

The Riemann sum can be thought of an approximation to the volume between the graph $z = f(x,y)$ & the x-y plane rectangle R, if f is positive.



$$\Delta V_k = \underbrace{f(x_k^*, y_k^*)}_{\text{height}}\ \underbrace{\Delta A_k}_{\text{base area}}$$

Example: Let $f(x,y) = x + y$
$R = \{(x,y): 0 \le x \le 2,\ 0 \le y \le 2\}$



Subdivide R into 4 unit squares as indicated.
Find two Riemann sums for f,
once using midpoints (•'s),
once using lower left-hand corners (■'s).
Which Riemann sum (do you guess) is better approximating the volume?

Theorem: If f is continuous on the rectangle $\mathcal{R} = \{(x,y) : a \le x \le b,\ c \le y \le d\}$

Then

$$\lim_{|P| \to 0} \sum_k f(x_k^*, y_k^*)\, \Delta A_k$$ exists. We write

$$\iint_{\mathcal{R}} f(x,y)\, dA$$ for its limit, and call it the (double) integral of f over $\mathcal{R}$.

the norm of P is defined to be the maximum length of all subrectangle diagonals

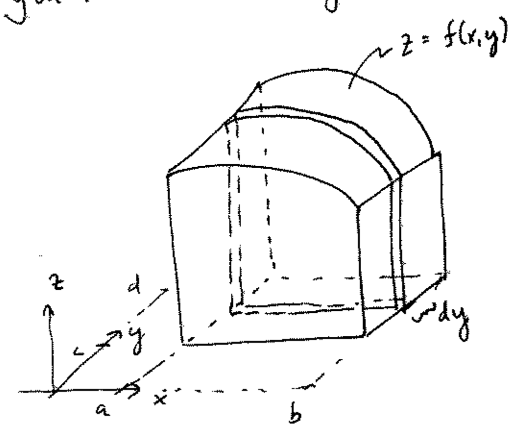
If f is positive the integral value is the volume between the graph $z = f(x,y)$ and the rectangle $\mathcal{R}$ in the x-y plane.
(If f changes sign you are computing a net volume, or signed volume, analogous to signed area in 1210.)

Theorem: Double integrals over rectangles can be computed as iterated single integrals:

If $\begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}$ on $\mathcal{R}$, then

$$\iint_{\mathcal{R}} f(x,y)\, dA = \int_a^b \Big(\underbrace{\int_c^d f(x,y)\, dy}_{\substack{x \text{ is const. in this integral} \\ \text{with respect to } y.}} \Big)\, dx = \int_c^d \Big(\underbrace{\int_a^b f(x,y)\, dx}_{\substack{y \text{ is const in this} \\ \text{integral.}}} \Big)\, dy$$

When f is positive this theorem is just a special case of how you found volumes by slicing in 1210:



cross-section area

$$dV = A(y)\, dy$$

$$V = \int_c^d A(y)\, dy$$

$$= \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

Examples

$0 \le x \le 2$
$0 \le y \le 2$
$f(x,y) = x+y.$

(see page 2).

$$\iint_R f(x,y)\,dA =$$

How does your answer compare to the Riemann sums on page 2?
Can you explain your answer geometrically?

Example $f(x,y) = \dfrac{y}{(xy+1)^2}$

$0 \le x \le 1$
$0 \le y \le 2$

Example $f(x,y) = e^{2y} + \cos x$

$0 \le x \le \pi$
$0 \le y \le 1$