Math 2210-4
Wed 2 March
§ 15.5 Directional derivatives and the gradient

Recall: $f(x,y)$ differentiable at (x_0,y_0) means it has a good tangent approximation there:

$$f(x_0+h_1, y_0+h_2) = f(x_0,y_0) + h_1 f_x(x_0,y_0) + h_2 f_y(x_0,y_0) + h_1\varepsilon_1(\vec{h}) + h_2\varepsilon_2(\vec{h}) \quad , \quad \varepsilon_1, \varepsilon_2 \to 0 \text{ as } \vec{h}\to\vec{0}$$

i.e.

$$f(x,y) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}_{T(x,y)} + \vec{h}\cdot\vec{\varepsilon}(\vec{h}) \qquad \vec{\varepsilon}(\vec{h})\to\vec{0} \text{ as } \vec{h}\to\vec{0}$$

and if we write this in vector notation, we see that the definition makes sense for a fcn of 3, 4, or n variables:

$$f(\vec{p}_0+\vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(\vec{h})$$

$\vec{\varepsilon}(\vec{h})\to\vec{0}$ as $\vec{h}\to\vec{0}$.

$\nabla f(\vec{p}_0)$ = the vector of partial derivs of f, at $\vec{p}_0$ called gradient of f at $\vec{p}_0$

Theorem
- If the partial derivs of f are continuous near $\vec{p}_0$, then f is differentiable at $\vec{p}_0$
- Theorem: f differentiable at $\vec{p}_0$ implies f continuous there. Converse not true

Reason

$$f(\vec{p}_0+\vec{h}) - f(\vec{p}_0) = \nabla f(p_0)\cdot\vec{h} + \vec{h}\cdot\vec{\varepsilon}(\vec{h})$$

$$\Rightarrow \quad |f(\vec{p}_0+\vec{h}) - f(\vec{p}_0)| \le | \qquad |$$

$$\le |\ | + |\ | = |\nabla f(p_0)||\vec{h}||\cos\theta_1| + |\vec{h}||\vec{\varepsilon}(\vec{h})||\cos\theta_2|$$

$$\le |\nabla f(p_0)||\vec{h}| + |\vec{h}||\vec{\varepsilon}(\vec{h})|$$

$$\to 0 \text{ as } \vec{h}\to\vec{0}.$$

Directional derivatives

$$f_x(x_0,y_0) = \lim_{\substack{h\to 0 \\ \nwarrow \text{ scalar}}} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h} = \text{rate of change of } f \text{ in } \hat{\imath} \text{ direction}$$

$$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0,y_0+h) - f(x_0,y_0)}{h} = \text{rate of change of } f \text{ in } \hat{\jmath} \text{ direction.}$$

Let $\vec{u}$ be a unit vector in <u>any</u> direction, f a function of several variables, differentiable at $\vec{p}_0$.

$$\boxed{D_{\vec{u}} f(\vec{p}_0) := \lim_{\substack{h\to 0 \\ \uparrow \\ \text{scalar}}} \frac{f(\vec{p}_0 + h\vec{u}) - f(\vec{p}_0)}{h}}$$

is called the directional derivative of f in direction $\vec{u}$, at $\vec{p}_0$

<u>Examples</u>: this includes partial derivatives as a special case:

$D_{\hat{\imath}} f(x_0,y_0) =$

$D_{\hat{k}} f(x_0,y_0,z_0) =$

$D_{\langle 0,0,0,1\rangle} f(x_0,y_0,z_0,w_0) =$

<u>Theorem</u>: Directional derivatives are easy to compute, if f is differentiable at at $\vec{p}_0$.

<u>proof</u>: Use the tangent approximation formula

$$f(\vec{p}_0 + h\vec{u}) = f(\vec{p}_0) + \nabla f(\vec{p}_0)\cdot h\vec{u} + h\vec{u}\cdot\vec{\varepsilon}(h\vec{u})$$

$$\frac{f(p_0 + h\vec{u}) - f(\vec{p}_0)}{h} = \frac{\nabla f(\vec{p}_0)\cdot \cancel{h}\vec{u}}{\cancel{h}} + \frac{\cancel{h}\vec{u}\cdot\vec{\varepsilon}(h\vec{u})}{\cancel{h}}$$

take $\lim_{h\to 0}$: $\boxed{D_{\vec{u}} f(\vec{p}_0) = \nabla f(\vec{p}_0)\cdot\vec{u}}$

Because $\vec{u}\cdot\vec{\varepsilon}(h\vec{u}) \le |\vec{u}||\vec{\varepsilon}| = |\vec{\varepsilon}|$

$-|\vec{u}||\vec{\varepsilon}| = -|\vec{\varepsilon}| \to 0$

$\searrow 0$ $\quad \swarrow 0$

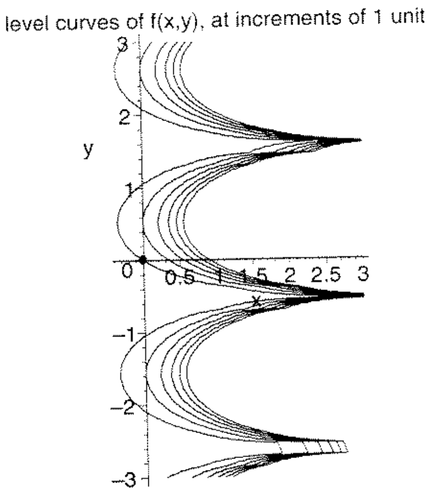
③

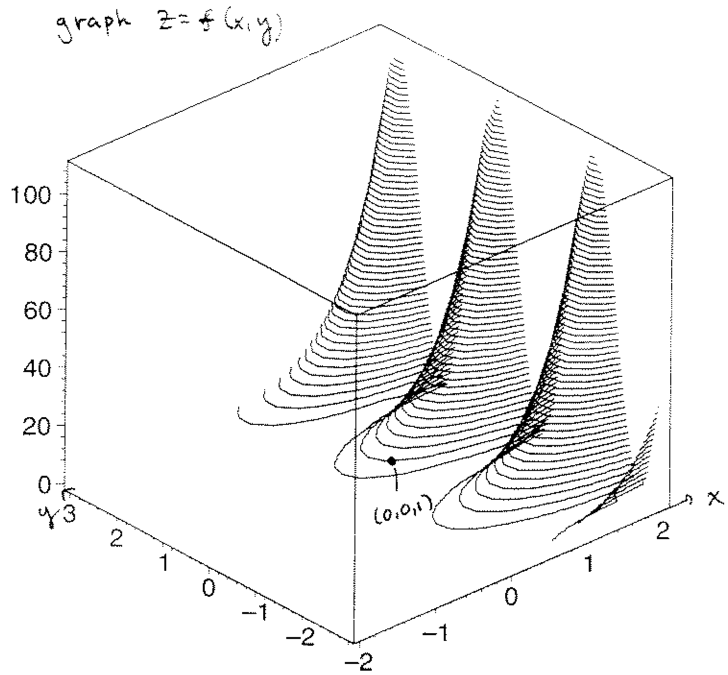
More fun with index cards

$$f(x,y) = e^{2x}(1+\sin 3y)$$

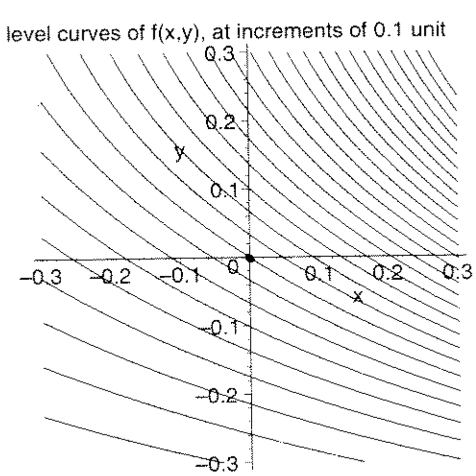
$$(x_0, y_0) = (0,0).$$

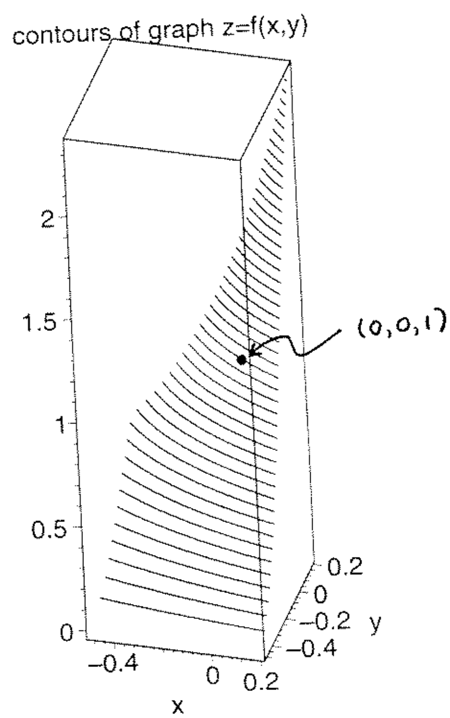
Far view





Medium view

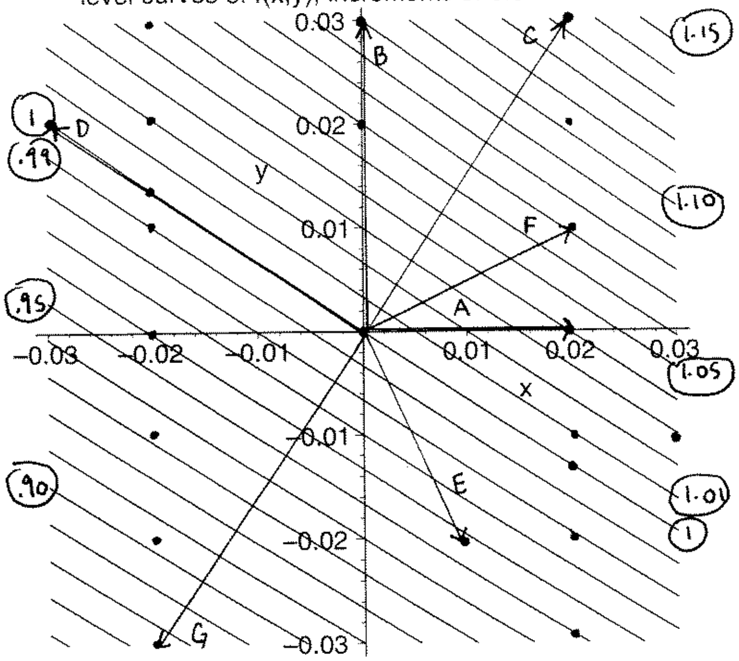


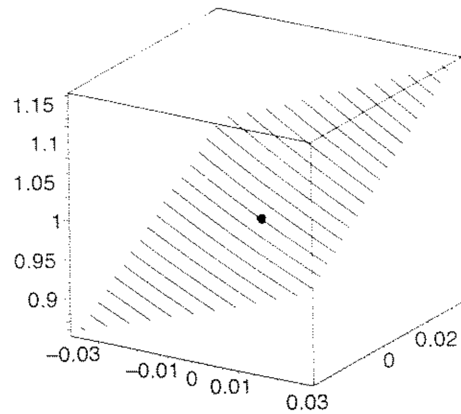


Near view

$f(x,y) = e^{2x} + e^{2x}\sin 3y$
near view

level curves of f(x,y), increments of 0.01 unit





Estimate, and find exact values, for $D_{\vec{u}}f(\vec{0})$: <u>Step 0</u>: Find $\nabla f(0,0)$.

A) $\vec{u} = \langle 1,0\rangle$ $\quad D_{\vec{u}}f(\vec{0}) \approx \dfrac{1.04-1}{.02} = 2$

$f_x(\vec{0}) = \nabla f(\vec{0})\cdot\langle 1,0\rangle = 2$!

B) $\vec{u} = \langle 0,1\rangle$ $\quad D_{\vec{u}}f(\vec{0}) \approx \dfrac{1.09-1}{.03} = 3$

$f_y(\vec{0}) = 3$!

C) $\vec{u} = \frac{1}{\sqrt{13}}\langle 2,3\rangle$

D) $\vec{u} = \frac{1}{\sqrt{13}}\langle -3,2\rangle$

E) $\vec{u} = \frac{1}{\sqrt{5}}\langle 1,-2\rangle$

F) $\vec{u} = \frac{1}{\sqrt{5}}\langle 2,1\rangle$

G) $\vec{u} = -\frac{1}{\sqrt{13}}\langle 2,3\rangle$

Conclusions

$$D_{\vec{u}} f(\vec{p}_0) = \nabla f(\vec{p}_0)\cdot\vec{u} = |\nabla f(\vec{p}_0)|\,|\vec{u}|\cos\theta = |\nabla f(p_0)|\cos\theta$$

So

$$-|\nabla f(\vec{p}_0)| \le D_{\vec{u}} f(p_0) \le |\nabla f(p_0)|$$

if $\vec{u} = -\dfrac{\nabla f(p_0)}{|\nabla f(p_0)|}$ get this min value

if $\vec{u} = \dfrac{\nabla f(p_0)}{|\nabla f(p_0)|}$ get this max value

gradient points in direction of maximum increase

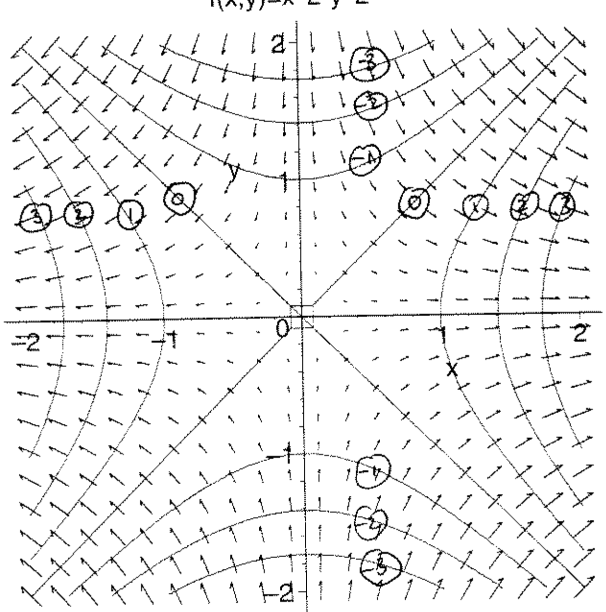
$D_{\vec{u}} f(p_0) = 0$ if $\vec{u}$ is in direction of level curve thru $\vec{p}_0$

i.e. $\cos\theta = 0$

i.e. $\nabla f(\vec{p}_0) \perp$ level curve

(level surface in higher dims!)

gradient and level curves for f(x,y)=x^2-y^2



My directional derivative calculations:

```
> Digits:=3;
                                        Digits := 3
> .04/.02;         #<1,0> approx
  2;               #exact
                                        2.00
                                        2
> .06/.02;         #<0,1> approx
  3;               #exact
                                        3.00
                                        3
> .133/.036;  #<2,3> approx
  sqrt(13.0);#exact
                                        3.69
                                        3.61
> .072/.022;   #<2,1> approx;
  7/sqrt(5.0);#exact
                                        3.27
                                        3.12
> 0/3.6;           #<-3,2> approx
  0;               #exact
                                        0.
                                        0
> -.01/.022;     #<-2,1> approx
  -1/sqrt(5.0);#exact
                                        -.455
                                        -.446
> (.875-1)/.0355;  #<-2,-3> approx
  -sqrt(13.0);#exact
                                        -3.52
                                        -3.61
> (.958-1)/.022    # <1,-2> approx     -1.91
  -4/√5                     exact      -1.79
```