

Cross product

Unlike the dot product, $\vec{u} \cdot \vec{v}$, in which the value is a scalar (i.e. number), the value of the cross product, $\vec{u} \times \vec{v}$, is a vector, obtained by a process involving determinants.

Digress to determinants

2x2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

these bars stand for determinant, not magnitude.

3x3 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} := a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

note minus sign
↓
det of "minor" matrix obtained by deleting row & column in which a_1 appears

minor det by deleting row & col of a_2

$$= a_1 b_2 c_3 + a_2 c_1 b_3 + a_3 b_1 c_2 - a_1 c_2 b_3 - a_2 c_3 b_1 - a_3 c_1 b_2$$

Where do determinants come from?

- algebra related to matrices ($\det \neq 0$ if and only if matrix has an inverse)
- geometry of areas & volumes (as we shall see for 2x2 & 3x3).

example (1a) Find the area of the parallelogram spanned by

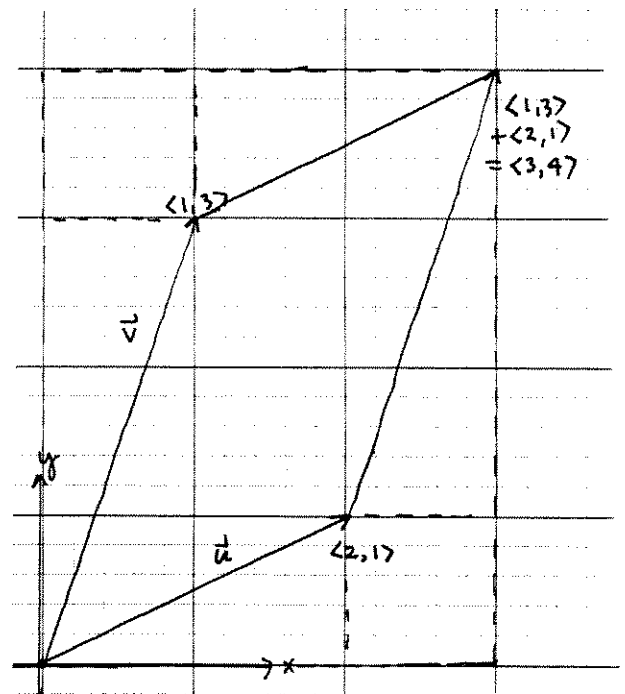
$$\vec{u} = \langle 2, 1 \rangle$$

$$\vec{v} = \langle 1, 3 \rangle$$

(1b) Compare to the two determinants

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} =$$



Fact: $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \pm \text{area of } \vec{u}, \vec{v} \text{ parallelogram}$ (+ if $\{\vec{u}, \vec{v}\}$ are right-handed, - if $\{\vec{u}, \vec{v}\}$ are left-handed)

(2)

$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \pm \text{volume of } \vec{u}, \vec{v}, \vec{w} \text{ parallelepiped}$ (+ if $\{\vec{u}, \vec{v}, \vec{w}\}$ are right-handed, - if $\{\vec{u}, \vec{v}, \vec{w}\}$ are left-handed).
(see page 4!)

end digress.

Cross product

$$\vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\begin{aligned} \vec{u} \times \vec{v} &:= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{k} \\ &= \langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \rangle \\ &= \langle u_2 v_3 - v_2 u_3, v_1 u_3 - u_1 v_3, u_1 v_2 - v_1 u_2 \rangle \end{aligned}$$

example (2)

Compute $\langle 2, 1, 0 \rangle \times \langle 1, 3, 0 \rangle$

Cross product algebra (some surprises!)

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. In particular, $\vec{u} \times \vec{u} = \vec{0}$!

2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
 $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$

4. $\vec{u} \times \vec{0} = \vec{0}$

5. $(\vec{u} \times \vec{v}) \cdot \vec{w} = \vec{u} \cdot (\vec{v} \times \vec{w})$; in fact $(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

7. $\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$

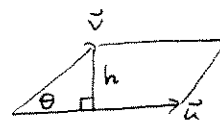


Theorem (Gives geometric meaning of cross product)

(A) $\vec{u} \times \vec{v}$ is $\perp$ to $\vec{u}$ and $\vec{v}$.

Also, $\{\vec{u}, \vec{v}, \vec{u} \times \vec{v}\}$ are right-handed

(B) $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin\theta$ = area of $\vec{u}, \vec{v}$ spanned by $\vec{u}, \vec{v}$



$$h = |\vec{v}|\sin\theta$$

$$\text{so } A = |\vec{u}|h = |\vec{u}||\vec{v}|\sin\theta.$$

(A) check: $\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1(u_2v_3 - v_2u_3) + u_2(v_1u_3 - u_1v_3) + u_3(u_1v_2 - v_1u_2) = 0$
 $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$ analogous.

(B) check: depends on Lagrange identity, which is "fun" algebra (see HW)
 $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$

Then,

$$= |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2\cos^2\theta$$

$$= |\vec{u}|^2|\vec{v}|^2(1 - \cos^2\theta)$$

$$= |\vec{u}|^2|\vec{v}|^2\sin^2\theta.$$

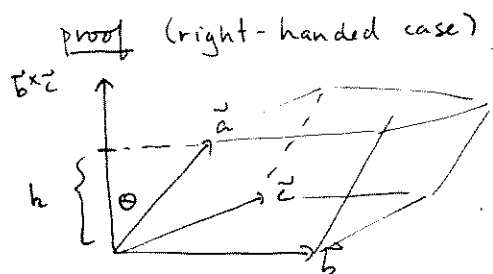
then take $\sqrt{\quad}$'s! ■

Example

(3) Check Theorem with example 2, $\langle 2, 1, 0 \rangle \times \langle 1, 3, 0 \rangle$.

Example (4) Recompute $\langle 2, 1, 0 \rangle \times \langle 1, 3, 0 \rangle = (2\hat{i} + \hat{j}) \times (\hat{i} + 3\hat{j})$ using algebra 1-7.

Theorem $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm \text{volume of parallelepiped determined by } \vec{a}, \vec{b}, \vec{c}$
 (+ vol if $\{\vec{a}, \vec{b}, \vec{c}\}$ right-handed, else - vol).



$$\begin{aligned} \text{Vol} &= (\text{area of base})(\text{height}) \\ &= |\vec{b} \times \vec{c}| |\vec{a}| \cos \theta \end{aligned} \quad \left(\text{in case } \cos \theta > 0, \text{ i.e. } \{\vec{a}, \vec{b}, \vec{c}\} \text{ right-handed.} \right.$$

else opposite.)

$$\text{but } \vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}| |\vec{b} \times \vec{c}| \cos \theta$$

$$\text{so } |\vec{a}| \cos \theta = \frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}$$

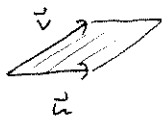
$$= \cancel{|\vec{b} \times \vec{c}|} \left(\frac{\vec{a} \cdot (\vec{b} \times \vec{c})}{\cancel{|\vec{b} \times \vec{c}|}} \right)$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) = \left| \begin{array}{c} \vec{a} \\ \vec{b} \\ \vec{c} \end{array} \right| \quad \blacksquare$$

Example 5 Interpret $\begin{vmatrix} 0 & 0 & 3 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix}$ as a volume.

What we can do with cross product :

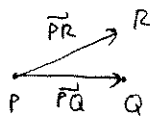
areas



volumes



plane eqns



$$\langle a, b, c \rangle = \vec{PQ} \times \vec{PR}$$

more!

Homework for Friday 1/28

14.3 3, (5), 9, (10) (11) 13, (15), (19), 21, (25) (26)

14.4 (5) (9) (17) (19) (28) (29a) 30a

14.6 (2) (5) (7) (8) (9) (14)