Math 2210-4
Friday 1/14

Recall geometric & algebraic
ways of defining
vectors
vector addition
scalar multiplication
magnitude

Finish Wed notes:
page 3–4 vector algebra properties
unit vectors
the standard basis vectors $\hat{\imath}\ \hat{\jmath}\ \hat{k}$

§13.3 ($\mathbb{R}^2$), 14.2 ($\mathbb{R}^3$):
Dot product (another algebraic operation on vectors with geometric meaning).

if $\vec{a} = \langle a_1, a_2, a_3 \rangle$
$\vec{b} = \langle b_1, b_2, b_3 \rangle$

then $\vec{a}\cdot\vec{b} := a_1b_1 + a_2b_2 + a_3b_3$
"$\vec{a}$ dot $\vec{b}$"

(In $\mathbb{R}^n$, $\vec{a}\cdot\vec{b} = \sum_{i=1}^{n} a_ib_i$)

① $\langle 2, 4\rangle \cdot \langle 3, -1\rangle =$

$\langle -1, 7, 4\rangle \cdot \langle 6, 2, -\tfrac{1}{2}\rangle =$

Dot product algebra:

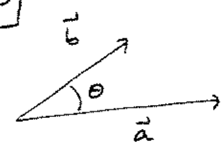
1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$ (so $|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$)
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
   $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$
4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$
   (c a scalar!)
5. $\vec{a}\cdot\vec{0} = 0$

② Check some of these algebra rules
(to help you become comfortable using them)

The reason dot products are useful is that they let us do geometry. (As we will see.)
The geometric meaning of dot product is contained in the formula

<u>Theorem</u>: $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$

(where $\measuredangle \vec{a},\vec{b} = \theta$:



Thus

(i) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

(ii) $\vec{a} \perp \vec{b}$ exactly if $\vec{a}\cdot\vec{b} = 0$

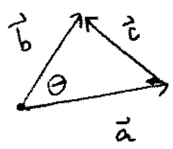
3 14.1 #7) (which was HW due today), worked with dot product instead of distance formula:

Show $P = (2,1,6)$
$Q = (4,7,9)$
$R = (8,5,-6)$

are vertices of a right triangle.

Why the theorem on page 2 is true:

Consider the triangle



We will compute $|\vec{c}|^2$ two ways, set the expressions equal, and deduce $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ (after canceling terms)

(i) Law of cosines:

$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$

← generalizes Pythagorean Theorem, and is actually a consequence of it, see HW.

(ii) Vector computation:

$$\vec{c} = \vec{b} - \vec{a}$$

(because, $\vec{a} + (\vec{b} - \vec{a}) = \vec{b}$!)

so $|\vec{c}|^2 = \vec{c}\cdot\vec{c} = (\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})$

$= \vec{b}\cdot(\vec{b}-\vec{a}) - \vec{a}\cdot(\vec{b}-\vec{a})$ (dot prod ~~use~~ algebra!)

$= \vec{b}\cdot\vec{b} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{a}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$

Thus (setting computation (i) = computation (ii))

$$\cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}||\vec{b}|\cos\theta = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2\vec{a}\cdot\vec{b}$$

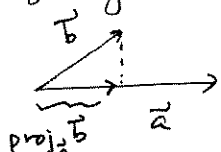
$$\cancel{-2}|\vec{a}||\vec{b}|\cos\theta = \cancel{-2}\,\vec{a}\cdot\vec{b}$$

$$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$$ ∎

On Wednesday (next class) we'll use dot product to understand

1) the graph of $ax + by + cz = d$ is a plane, with $\perp$ vector $\langle a, b, c\rangle$

2) how to project one vector orthogonally onto another, using only algebra

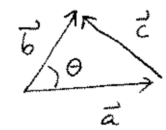


3) Compute "work" when moving against constant force fields.

Homework for Friday 1/21 : As always, circled problems hand in, others recommended.

1. Verify the law of cosines

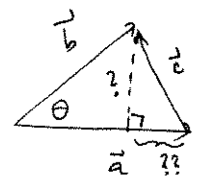
$$|\vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$



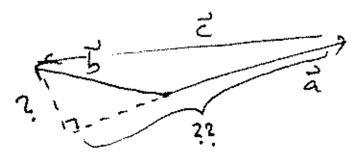
using trig, as indicated.

(a) $\theta = 0, \pi/2, \pi$. (These are special cases; $\theta = \pi/2$ is just Pyth, so you don't need to redo it, after last HW)

(b) $0 < \theta < \pi/2$. Label so that $|\vec{b}| \le |\vec{a}|$. Find lengths ?, ?? from trig. Then apply Pyth. thm.



(c) $\pi/2 < \theta < \pi$. Follow same steps as in (b):



Book problems

13.2 2, 3, 4, 7, 9, 12, 17, 19

13.3 1 a, c, e, 2 b, d, f, 3b, 4c, 5a, 14, 15, 16, 23, 27, 28, 30, 33a

14.2 1a (draw box with pts at opposite corner, to help see)
3a, 5, 7, 10, 11, 12, 13, 15, 18, 23, 25a, 26, 29, 31, 33, 40, 43, 45, 50