

Math 2210-4  
February 7, 2005  
Curvature, Speed and the Roller Coaster Equation

***Note:** I made this handout with MAPLE, a computer software program which is available on our Math Department computers, as well as on most University computers (e.g. Engineering labs and Marriott) around campus. MAPLE does any computation you could want in undergraduate mathematics, and draws any mathematical picture you could need, if you learn how to ask. Combining these mathematical tools with MAPLE's modest word processing capabilities, you can create documents like this one. Alternately, if you need to create a scientific report, you can use another word processor, like MS Word, and paste MAPLE output into it. You can explore MAPLE by going to our Math Student Center in the basement - where all students enrolled in math classes have special math accounts, or by going to the Marriott computers, where you have your (different) U-accounts. There is a "New Users Tour" which you can take once you open MAPLE.*

*If there is sufficient interest as the semester progresses, I can arrange a time outside of class to help you learn about MAPLE in our computer classroom LCB 115. Those of you moving on to Math 2250 or to Math 2270-2280 will experience more MAPLE in those classes.*

We continue our discussion from Friday. We were on page 3 of those notes, and had defined curvature in terms of the magnitude of the rate of change of the unit tangent vector, with respect to arclength. We had also checked that the curvature of a circle of radius  $A$  is  $1/A$ . Please refer to pages 3-4 of Friday's notes for further discussion and pictures related to the following recap and continuation. By the way, what follows is an excellent sequence of definitions and derivations for you to learn and understand, because of the content and also because we use almost all of the important vector Calculus ideas we have talked about so far. The text has a similar discussion on pages 614-616.

Let  $\mathbf{r}(t)$  be the position vector of a parametric curve,  $a \leq t \leq b$ , with  $\mathbf{r}'(t)$  never zero. (I will underline vectors in MAPLE). Let  $\mathbf{r}'(t) = \mathbf{v}(t)$  be the tangent (velocity) vector, and let  $v(t) = |\mathbf{v}(t)|$  be its magnitude (speed). In our discussion we use the arclength function  $s(t)$  obtained by integrating the **speed** between  $a$  and  $t$ :

$$s(t) = \int_a^t v(t) dt$$

Note that because  $s(t)$  is increasing it has an inverse function  $t(s)$ . Furthermore, at corresponding points we can use the chain rule on the inverse function identity  $t(s(t))=t$  (like you did in Calculus when you considered the natural logarithm and exponential functions), to deduce in this case that

$$\frac{dt}{ds} = \frac{1}{\left[ \frac{ds}{dt} \right]} = \frac{1}{v}.$$

We define the unit tangent vector  $\mathbf{T}(t)$  by normalizing the velocity vector:

$$\mathbf{T}(t) = \mathbf{r}'(t) / v.$$

The derivative of  $\mathbf{T}$  with respect to arclength  $s$  is a geometric way to measure how much the range curve is bending at the point with position vector  $\mathbf{r}(t)$ . The magnitude of this vector is what we called the curvature  $\kappa(t)$ :

$$\kappa(t) = |d\mathbf{T}/ds| = |d\mathbf{T}/dt| (dt/ds) = |d\mathbf{T}/dt| / v.$$

It was with this formula that we computed the curvature of a radius  $A$  circle, and deduced that its value was  $1/A$ . For curves which are not already parameterized at constant speed, the formula above turns out to be cumbersome to work with. We can derive a better formula for  $\kappa(t)$  as a consequence of the "roller coaster" equation below, which decomposes a curve's acceleration vector into tangential and normal parts. To derive this equation, first notice that since  $|\mathbf{T}(t)|$  is identically one,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1.$$

(Maple doesn't have a good dot product symbol.) Taking the  $t$ -derivative of this identity, with the product rule for dot product, deduce

$$2 \mathbf{T}'(t) \cdot \mathbf{T}(t) = 0.$$

Thus  $\mathbf{T}'(t)$  is perpendicular to  $\mathbf{T}(t)$ . Note from the defining equation of  $\kappa(t)$  above that the magnitude of  $\mathbf{T}'(t)$  is the product  $v(t) \kappa(t)$ . We call the unit vector in the  $\mathbf{T}'(t)$  direction the unit normal  $\mathbf{N}(t)$ . Thus  $\mathbf{N}(t) = \mathbf{T}'(t) / (v(t) \kappa(t))$ . We rewrite this as

$$\mathbf{T}'(t) = v(t) \kappa(t) \mathbf{N}(t).$$

We can now derive the roller coaster equation. Use the definition of  $\mathbf{T}(t)$  to write

$$\mathbf{r}'(t) = v(t) \mathbf{T}(t).$$

Taking  $t$ -derivatives, we use the product rule for a scalar function times a vector function, and our computation for  $\mathbf{T}'(t)$  to deduce

$$\mathbf{r}''(t) = v'(t) \mathbf{T}(t) + v(t) \mathbf{T}'(t)$$

$$\mathbf{r}''(t) = v'(t) \mathbf{T}(t) + \kappa(t) v(t)^2 \mathbf{N}(t).$$

This is a great equation, which we shall call the roller coaster equation (RCE).

**Application I:** better formulas for computing curvature. Consider the cross product  $\mathbf{r}'(t) \times \mathbf{r}''(t)$ , use (RCE), distribute the cross product over the addition on the right side of this formula, and factor the scalar functions out:

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= v'(t) \mathbf{r}'(t) \times \mathbf{T}(t) + \kappa(t) v(t)^2 \mathbf{r}'(t) \times \mathbf{N}(t) \\ &= \kappa(t) v(t)^2 \mathbf{r}'(t) \times \mathbf{N}(t),\end{aligned}$$

because  $\mathbf{r}'(t)$  and  $\mathbf{T}(t)$  are parallel. Now take the magnitude of this vector identity. Note that since  $\mathbf{r}'(t)$  and  $\mathbf{N}(t)$  are perpendicular, the magnitude  $|\mathbf{r}'(t) \times \mathbf{N}(t)|$  is simply  $v(t)$ . Thus

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \kappa(t) v(t)^3$$

which we rearrange to

$$\kappa(t) = |\mathbf{r}'(t) \times \mathbf{r}''(t)| / v(t)^3.$$

**Special cases:** (check!!!) If  $\mathbf{r}(t) = \langle x(t), y(t), 0 \rangle$  is the position vector of a curve in the x-y plane, then this formula yields

$$\kappa(t) = |x'(t) y''(t) - x''(t) y'(t)| / v(t)^3$$

If, alternately, this parametric plane curve lies on a graph  $y=f(x)$ , then we can choose to re-parameterize it by  $\rho(t) = \langle t, f(t) \rangle$ , yielding (after we substitute  $t=x$  for the parameterization  $\rho$  :

$$\kappa(x) = |f''(x)| / (1 + f'(x)^2)^{(3/2)}.$$

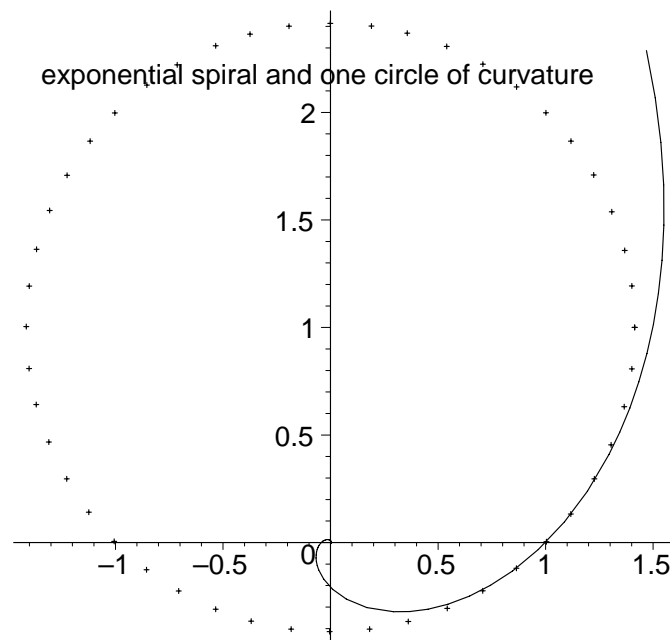
You will use this last formula in some of your homework problems. In the following example it will be more convenient to keep the parameterization we are given and use parametric formula.

**Example 1:** Consider the exponential spiral curve with position vector

$$\mathbf{r}(t) = \langle e^t \cos(t), e^t \sin(t) \rangle.$$

Find  $\mathbf{T}$ ,  $\mathbf{N}$ , and the curvature of this curve at the point where  $t=0$ . Then find the center and radius of the "circle of curvature" at this point - namely the circle passing through this point, and having the same curvature and tangent and normal vectors as  $\mathbf{r}(t)$ , at that point. There's a picture of the (range) curve, for  $t$  between -10 and 1, as well as the circle of curvature, to help you out.

$$\mathbf{r}(t) = \langle \mathbf{e}^t \cos(t), \mathbf{e}^t \sin(t) \rangle$$



## **Application II:** Physics.

$$\mathbf{r}''(t) = v'(t) \mathbf{T}(t) + \kappa(t) v(t)^2 \mathbf{N}(t).$$

Even the special case of the roller-coaster equation, for planar motion around a circle, is important. If the speed of the rotating object is constant then there is no tangential component of acceleration. We deduce in this case that (centripetal) acceleration back towards the center of the circle has magnitude proportional to the speed squared, divided by the circle radius. Since Newton's Law says acceleration is proportional to force ( $\mathbf{F} = m\mathbf{a}$ ), this formula quantitatively describes the difficulty of making objects rotate in circles, depending on how fast they are going.

The roller coaster equation may have been Newton's key to finding his complete model of planetary motion. In the special case of circular orbits, Kepler's observed 3rd law of planetary motion is that the square of the planet period  $P$  is proportional to the cube of the radius  $R$ , with the same proportionality constant for all planets orbiting our sun. In other words,

$$P = C R^{(3/2)}.$$

On the other hand, the period is the total distance traveled in one orbit, divided by the speed:

$$P = \frac{2\pi R}{v}.$$

Setting these two expressions for the period equal and solving for the speed  $v$  yields

$$v = \frac{2\pi}{C\sqrt{R}}.$$

Substituting this value of  $v$  into the RCE, and using  $\kappa = \frac{1}{R}$ , we deduce the inverse square law of planetary acceleration:

$$|\mathbf{r}''(t)| = \frac{4\pi^2}{C^2 R^2}.$$

$$\mathbf{r}''(t) = v'(t) \mathbf{T}(t) + \kappa(t) v(t)^2 \mathbf{N}(t).$$

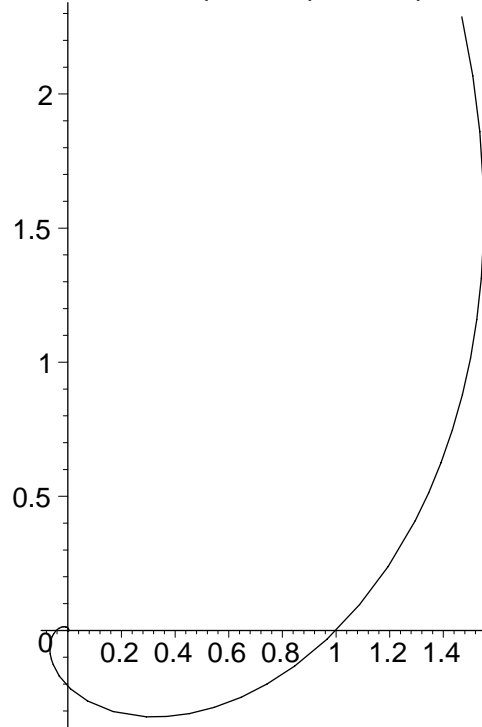
The roller coaster equation is interesting for every vector valued function  $\mathbf{r}(t)$ , as illustrated by this continuation of Example 1:

**Example 2:** Consider the same curve as in Example 1, namely

$$\mathbf{r}(t) = \langle e^t \cos(t), e^t \sin(t) \rangle.$$

- 2a) Compute  $\mathbf{r}''(0)$  and draw it carefully onto the graph of the range of  $\mathbf{r}(t)$ .
- 2b) Use a straight-edge to decompose  $\mathbf{r}''(0)$  into a sum of vectors in the  $\mathbf{T}$  and  $\mathbf{N}$  directions, at  $t=0$ . Include the  $\mathbf{T}$  and  $\mathbf{N}$  vectors in the picture, using a length ruler from the axes to insure that you actually draw unit vectors. Find the components of  $\mathbf{r}''(0)$  in the  $\mathbf{T}$  and  $\mathbf{N}$  directions, using your ruler.
- 2c) Compute the exact components of  $\mathbf{r}''(0)$  in the  $\mathbf{T}$  and  $\mathbf{N}$  directions, using the dot product (and/or geometry). Compare with the numerical values you got in (2b).
- 2d) Use instead the RCE formulas to compute the  $\mathbf{T}$  and  $\mathbf{N}$  components of  $\mathbf{r}''(0)$ . Compare with (2c).

acceleration decomposition problem problem



Extension of problem 36, section 13.5 (This is homework to be handed in on Friday):  
We consider the parametric curve

$$\mathbf{r}(t) = \langle t^2, t \rangle.$$

36a) Show this curve lies on the parabola with equation  $x = y^2$ .

36b) Compute the point with position vector  $\mathbf{r}(1)$ . Then Compute  $\mathbf{r}'(1)$ ,  $\mathbf{r}''(1)$ . Plot the point and these vectors appropriately and carefully onto the picture of the range curve below.

36c) Using a ruler which you construct as we did on the earlier example, draw a picture which decomposes  $\mathbf{r}''(1)$  into its tangential and normal pieces. Measure the length of each piece to determine numerical values of the tangential and normal components of acceleration.

36d) Find the unit tangent and normal vectors  $\mathbf{T}(1)$ ,  $\mathbf{N}(1)$ , analytically. Then use the dot product to compute the components of acceleration in these two directions. Compare with the numerical values in (36c).

36e) Use the RCE equation to recompute the components in (36d), by computing  $\kappa(1)$ ,  $v(1)$ ,  $v'(1)$ . You should get the same answers!

