Math 2210-4
Friday 4 Feb.

HW for Fri 11 Feb:

14.5 # 15, 23, carried over from last week, with sketches

13.4 16, 17, 21, 23, 31, 38 41

13.5 10 17 20 36 (36 will be extended Monday)

52 53 55

14.5 28 36 52 64

Using Calculus to understand the shape of curves, and the physics of particle motion.

Recall $\vec{r}(t)$, $\vec{r}\,'(t)$, $\vec{r}\,''(t)$

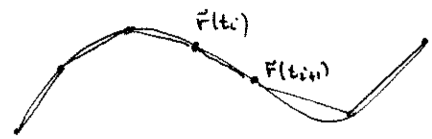
Curve length

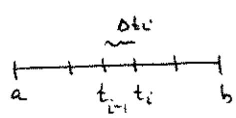
$\vec{r}:[a,b] \to \mathbb{R}^n$

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

Makes sense, since for constant speed, dist = (speed)(time)

Math reason: See § 6.4 p. 295 for details:





$\vec{r}(t_{i+1}) - \vec{r}(t_i)$

$= \langle f(t_{i+1}) - f(t_i),\ g(t_{i+1}) - g(t_i) \rangle$

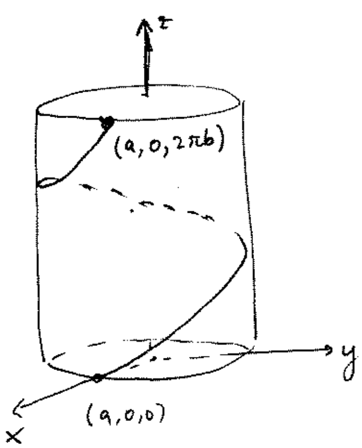
$= \langle f'(t_i^*)\Delta t_i,\ g'(t_i^{**})\Delta t_i \rangle$

$|\vec{r}(t_{i+1}) - \vec{r}(t_i)| = \sqrt{f'(t_i^*)^2 + g'(t_i^{**})^2}\,\Delta t_i$

Now take Riemann sums and limits

example 1 Consider one loop of the helix $\vec{r}(t) = \langle a\cos t,\ a\sin t,\ bt \rangle$, $0 \le t \le 2\pi$

Find its length



The

Antidifferentiation is important too!

Example 2 A projectile is thrown with an initial speed of v_0 m/s at an angle θ radians with the horizontal. Assuming only gravitational acceleration, describe its motion

$\vec{a} = \langle 0, -g \rangle$

$\vec{v}_0 = v_0 \langle \cos\theta, \sin\theta \rangle$

To proceed, we will need to use differentiation rules for vector-valued functions. These follow from differentiation rules for scalar functions

1. $D_t(\vec{F}(t) + \vec{G}(t)) = \vec{F}'(t) + \vec{G}'(t)$
2. $D_t(c\vec{F}(t)) = c\vec{F}'(t)$
3. $D_t(h(t)\vec{F}(t)) = h'F + hF'$
4. $D_t(\vec{F}(t) \cdot \vec{G}(t)) = \vec{F}' \cdot \vec{G} + \vec{F} \cdot \vec{G}$
5. $D_t(\vec{F}(t) \times \vec{G}(t)) = \vec{F}' \times \vec{G} + \vec{F} \times \vec{G}'$

(3–5: product rules)

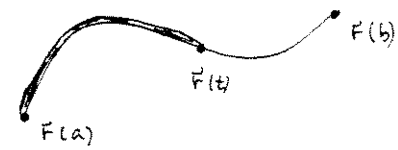
6. $D_t(\vec{F}(g(t)) = \vec{F}'(g(t))g'(t)$ } Chain rule.

Curvature of curves

$\vec{r}(t)$, $a \le t \le b$ with $\vec{r}\,'(t) \ne 0$.

Consider the arclength function $s(t)$,

$$s(t) = \int_a^t |\vec{r}\,'(\tau)|\,d\tau$$ = length of the curve arc, for $a \le \tau \le t$



since $\frac{ds}{dt} = |\vec{r}\,'(t)| > 0$

$s(t)$ has an inverse fcn $t(s)$,

and $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{v}$ (we continue to use scalar v for speed, $v = |\vec{v}|$).

Unit tangent vector $\vec{T}(t) = \frac{1}{v}\vec{r}\,'(t)$

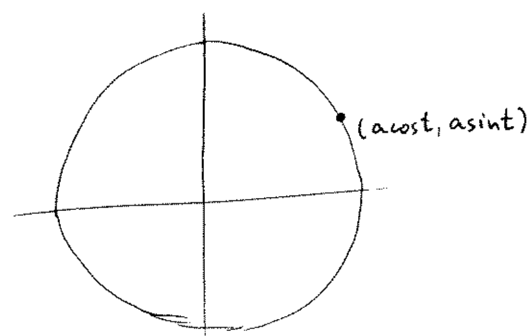
Curvature ("K") measures the rate of change of $\vec{T}$ with respect to arclength s

$$\boxed{K := \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\right|\frac{dt}{ds} = \frac{1}{v}\left|\frac{d\vec{T}}{dt}\right|}$$

Example 3: A circle of radius a has curvature $\frac{1}{a}$

Check this!

$\vec{r}(t) = \langle a\cos t, a\sin t\rangle$



$\vec{r}\,'(t) =$

$v =$

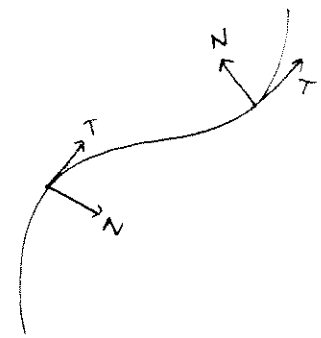
$\vec{T} =$

$\vec{T}\,'(t) =$

$K =$

Example 4 Can you find the curvature K for the helix $\vec{r}(t) = \langle a\cos t, a\sin t, bt\rangle$ in example 1?

It turns out there are better formulas
for computing κ (see page 585, 616).
They can be derived from a very important physics equation ...
("roller coaster eqtn")
see box below



Unit normal $\vec{N}$:

$$\vec{T}(t)\cdot\vec{T}(t) \equiv 1$$

$$\frac{d}{dt}: \quad 2\,\vec{T}(t)\cdot\vec{T}'(t) \equiv 0$$

$$\vec{N} := \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1}{\kappa v}\vec{T}'(t)$$ unit normal to curve

(defined when $\kappa \neq 0$)

$$\frac{\vec{r}'(t)}{v} = \vec{T}$$

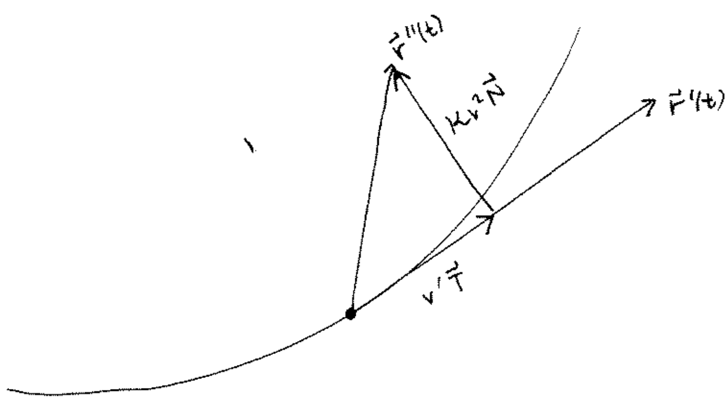
$$\vec{r}'(t) = v\vec{T}$$

$$\vec{r}''(t) = v'\vec{T} + v\vec{T}'(t)$$

$$\boxed{\vec{r}''(t) = v'\vec{T} + \kappa v^2\vec{N}}$$

the tangential component of acceleration indicates whether you're speeding up or slowing down

the normal component of acceleration is proportional to κ, and to the square of speed.



You can peel of the formula $\boxed{\kappa = \frac{|\vec{r}' \times \vec{r}''|}{v^3}}$ by taking $\vec{r}' \times$ the boxed eqtn above!

for a plane curve this reduces to $\kappa = \dfrac{|x'y'' - x''y'|}{(x'^2+y'^2)^{3/2}}$.