

§ 15.4 Differentiability.

What should it mean for a function

$f(x, y)$ to be differentiable at (x_0, y_0) ?

$f(x, y, z)$ " " (x_0, y_0, z_0) ?

Need a definition which is useful for Calculus to follow.

Return to our roots; 1-variable calculus

① $f(x)$ is differentiable at x_0 means

$$* \lim_{h \rightarrow 0} \frac{f(x_0+h) - f(x_0)}{h} =: f'(x_0) \text{ exists}$$

→ this definition won't work if
 $\vec{x}_0, \vec{h}$ are vectors (can't divide by a vector!)

→ reformulate def'n to get rid of denom:

* is equivalent to

$$\frac{f(x_0+h) - f(x_0)}{h} = m + \varepsilon(h)$$

where $m = f'(x_0)$

and $\varepsilon(h) \rightarrow 0$ as $h \rightarrow 0$

i.e.

$$f(x_0+h) - f(x_0) = mh + h\varepsilon(h)$$

denominator
gone! →

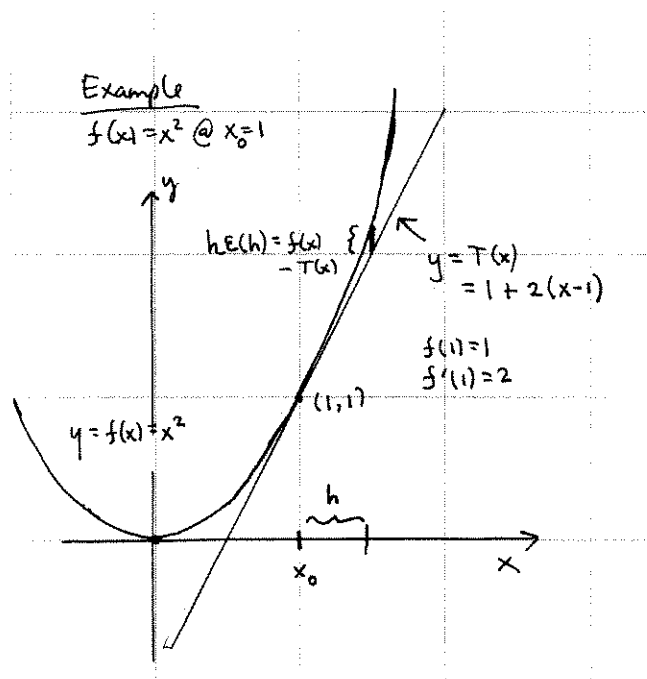
$$f(x_0+h) = f(x_0) + mh + h\varepsilon(h), \quad \varepsilon(h) \rightarrow 0 \text{ as } h \rightarrow 0$$

write $x = x_0 + h$:
 $x - x_0 = h$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + h\varepsilon(h)$$

Tangent
approx $T(x)$
to $f(x)$

Error
between
 $f(x)$ & $T(x)$,
 $\rightarrow 0$ faster
than h , since
 $\varepsilon(h) \rightarrow 0$,
as $h \rightarrow 0$.



Thus

- (2) $f(x, y)$ is differentiable at (x_0, y_0) means
 f has a good linear approximation (i.e. graph is a plane) there:

$$f(x_0+h_1, y_0+h_2) = f(x_0, y_0) + f_x(x_0, y_0)h_1 + f_y(x_0, y_0)h_2 + h_1\varepsilon_1(\vec{h}) + h_2\varepsilon_2(\vec{h})$$

where $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $\vec{h} \rightarrow 0$

Looks better in vector notation

$$\vec{p}_0 := (x_0, y_0)$$

$$\vec{h} := (h_1, h_2)$$

$$\nabla f := \langle f_x, f_y \rangle$$

$$\vec{\varepsilon}(\vec{h}) := \langle \varepsilon_1(h), \varepsilon_2(h) \rangle$$

$$f(\vec{p}_0 + \vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{h} \cdot \vec{\varepsilon}(\vec{h})$$

$$\vec{\varepsilon}(\vec{h}) \rightarrow 0 \text{ as } \vec{h} \rightarrow 0$$

- just like page 1
- works in any dimension

rewrite, if you wish

$$\begin{aligned} \langle x_0, y_0 \rangle &= \vec{p}_0 \\ \langle x, y \rangle &= \vec{p}_0 + \vec{h} \end{aligned}$$

$$f(x, y) = \underbrace{f(x_0, y_0) + \nabla f(x_0, y_0) \cdot \langle x - x_0, y - y_0 \rangle}_{:= T(x, y)} + \underbrace{\vec{h} \cdot \vec{\varepsilon}(\vec{h})}_{\text{error}}$$

the tangent approx to f at (x_0, y_0)
 (its graph $z = T(x, y)$ is tangent
 plane to graph $z = f(x, y)$, at $(x_0, y_0, f(x_0, y_0))$)

Example Find the
 tangent approximation to

$$f(x, y) = 4 - x^2 - y^2 \text{ at } (x_0, y_0) = (0, 0)$$

Sketch the graph

$$z = f(x, y)$$

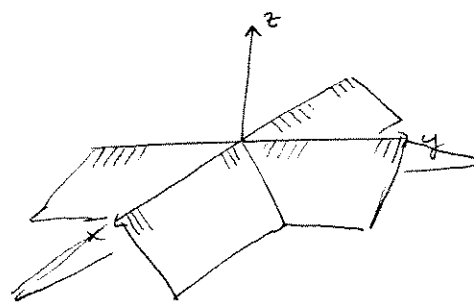
and its tangent plane

$$z = T(x, y)$$

Example Consider the "rooftop" function

$$f(x,y) = -(\text{minimum of } |x|, |y|)$$

$$= -(\text{distance from } (x,y) \text{ to the } x \text{ or } y \text{ axis})$$



graph $z = f(x,y)$

Do $f_x(0,0)$, $f_y(0,0)$ exist?

Is $f(x,y)$ differentiable at $(0,0)$?

Luckily, life is usually good:

Theorem If f_x and f_y are continuous functions near (x_0, y_0) , then f is differentiable at (x_0, y_0)

Explanation: you can understand this using the mean value theorem, from 12.10. Consult picture at right:

$$f(x_0+h_1, y_0+h_2) - f(x_0+h_1, y_0) = \left(\frac{\partial f}{\partial y}(x_0+h_1, d) \right) h_2$$

$$f(x_0+h_1, y_0) - f(x_0, y_0) = \left(\frac{\partial f}{\partial x}(c, y_0) \right) h_1$$

add

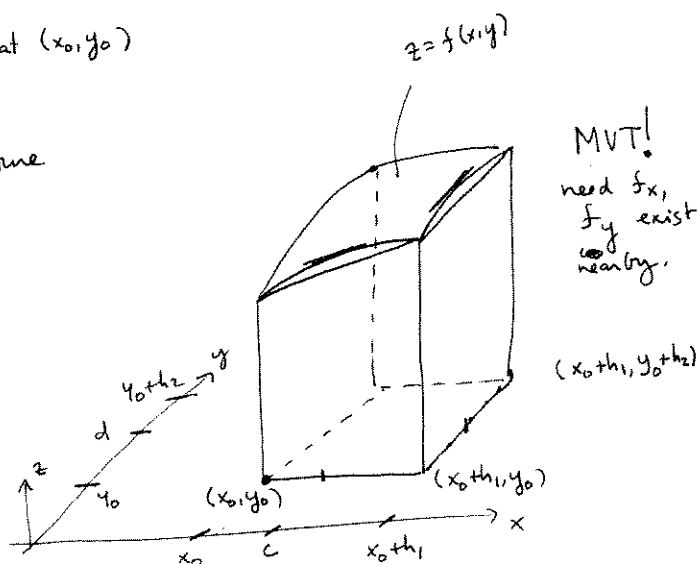
$$f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0) = \langle f_x(c, y_0), f_y(x_0+h_1, d) \rangle \cdot \vec{h}$$

$$f(\vec{p}_0 + \vec{h}) = f(\vec{p}_0) + \nabla f(\vec{p}_0) \cdot \vec{h}$$

$$+ \underbrace{\langle f_x(c, y_0) - f_x(x_0, y_0), f_y(x_0+h_1, d) - f_y(x_0, y_0) \rangle}_{\epsilon_1} \cdot \vec{h}$$

$$+ \underbrace{\langle f_x(x_0, y_0), f_y(x_0+h_1, d) - f_y(x_0, y_0) \rangle}_{\epsilon_2} \cdot \vec{h}$$

if f_x, f_y are continuous near $\vec{p}_0$
then $\epsilon_1, \epsilon_2 \rightarrow 0$ as $\vec{h} \rightarrow \vec{0}$



Example Where is the rooftop function differentiable?

Example : Was $f(x,y) = 4 - x^2 - y^2$ differentiable at $(0,1)$?
Why is this $f(x,y)$ differentiable everywhere?

Example : Show $f(x,y,z)$ is differentiable everywhere, $f(x,y,z) = e^x \sin 2y + 3z + 1$

(let $(x_0, y_0, z_0) = (0, 0, 0)$).

Find the linear (tangent) approximation to f at this point.

Compare $f(x,y,z)$ to $T(x,y,z)$

at $x = .1$
 $y = -.1$
 $z = .1$

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> f:=(x,y,z)->exp(x)*sin(2*y)+3*z+1;
                                f:=(x,y,z)->e^x sin(2y)+3z+1
> f(.1,-.1,.1);
f(.01,-.01,.01);
```

1.080436433

1.009800343

at $x = .01$
 $y = -.01$
 $z = .01$

Theorem : If f is differentiable at $\vec{p}_0$
then f is continuous at $\vec{p}_0$

(Converse not true: see roof function)

Reason : $f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0) = \nabla f(\vec{p}_0) \cdot \vec{h} + \vec{\varepsilon}(\vec{h}) \cdot \vec{h}$

so $|f(\vec{p}_0 + \vec{h}) - f(\vec{p}_0)| \leq |\nabla f(\vec{p}_0)| |\vec{h}| + |\vec{\varepsilon}(\vec{h})| |\vec{h}| \rightarrow 0$ as $\vec{h} \rightarrow 0$

so $\lim_{\vec{p} \rightarrow \vec{p}_0} f(\vec{p}) = f(\vec{p}_0)$ ■