Math 2210-4
Wed 2 Feb.
Calculus for parametric curves §13.4, 14.5
parametric curves with position vector

$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \longleftrightarrow$ in x-y plane

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle \quad \longleftrightarrow$ in $\mathbb{R}^3$

$$\boxed{\vec{r}\,'(t) := \lim_{\Delta t \to 0} \underbrace{\left(\tfrac{1}{\Delta t}\right)\left(\vec{r}(t+\Delta t) - \vec{r}(t)\right)}}$$ definition.

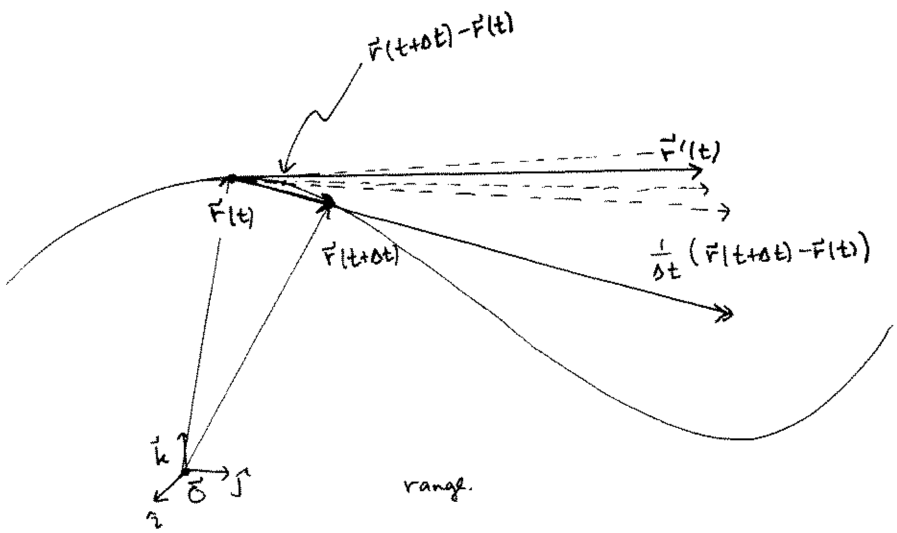
We call $\vec{r}\,'(t)$ the velocity vector (if t is time) or tangent vector

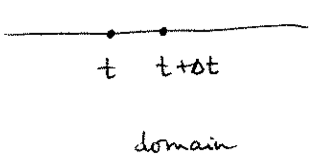
$$\frac{1}{\Delta t}\Big[f(t+\Delta t)\,\hat{\imath} + g(t+\Delta t)\,\hat{\jmath} + h(t+\Delta t)\,\hat{k} - \left(f(t)\,\hat{\imath} + g(t)\,\hat{\jmath} + h(t)\,\hat{k}\right)\Big]$$

$$= \lim_{\Delta t \to 0} \left(\frac{f(t+\Delta t) - f(t)}{\Delta t}\right)\hat{\imath} + \left(\frac{g(t+\Delta t) - g(t)}{\Delta t}\right)\hat{\jmath} + \left(\frac{h(t+\Delta t) - h(t)}{\Delta t}\right)\hat{k}$$

$$= f'(t)\,\hat{\imath} + g'(t)\,\hat{\jmath} + h'(t)\,\hat{k}$$

$$\boxed{\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle}$$ computation





geometric meaning

Example 1

For $\vec{r}(t) = \langle 2+t, 1+t^2 \rangle$

(which was page 1 Monday)

Compute $\dfrac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$

at $t = -1$

for $\Delta t = .1$

$\Delta t = -.1$

Compute $\vec{r}\,'(t)$

$\vec{r}\,'(-1)$

Illustrate your computations

Math 2210 work

```
> restart:
> r:=t->[2+t,1+t^2];
  (r(-0.9)-r(-1))/.1;
  (r(-1.01)-r(-1))/(-.01);
  diff(r(t),t);
  subs(t=-1,diff(r(t),t));
```

$$r := t \rightarrow [2+t, 1+t^2]$$

$$[1., -1.9]$$

$$[1., -2.01]$$

$$[1, 2\,t]$$

$$[1, -2]$$

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> with(plots):
Warning, the name changecoords has been redefined
> plot([2+t,1+t^2,t=-2..2],color=black,scaling=constrained);
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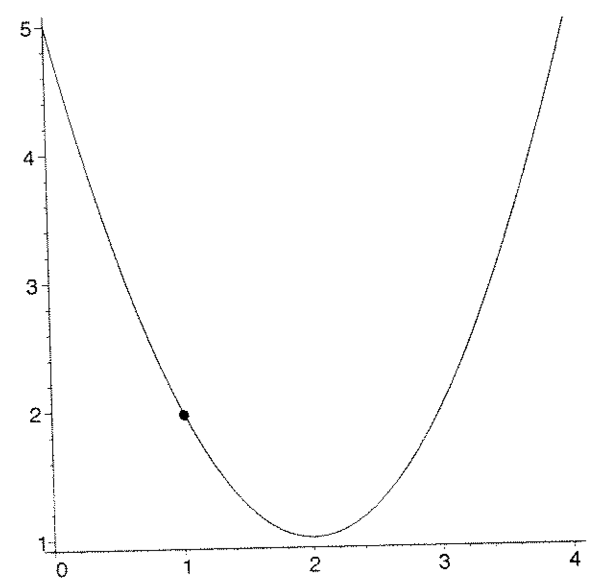


Example 2 For a curve in the plane,

$\vec{r}(t) = \langle f(t), g(t) \rangle$, lying on a graph $y = F(x)$

can you draw a picture illustrating

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad ?$$

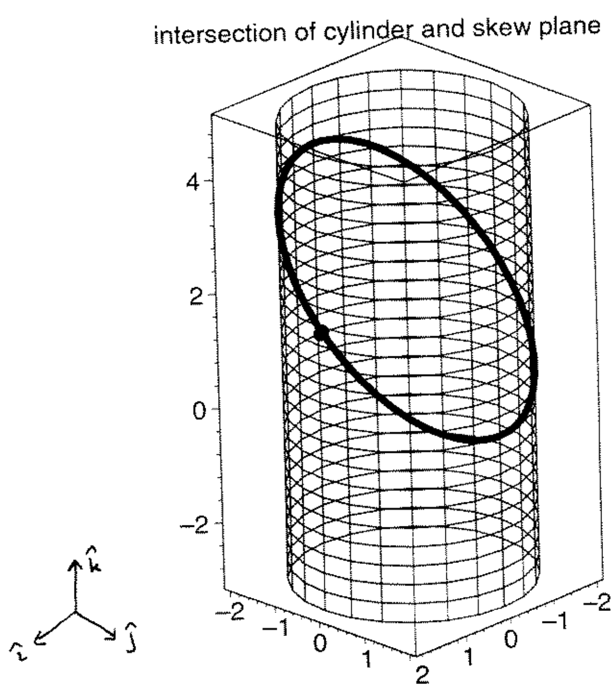
Example 3 $\vec{r}(t) = \langle 2\cos t, 2\sin t, 2-2\sin t \rangle$

$0 \le t \le 2\pi$

parameterized the intersection of the cylinder $x^2+y^2=4$ and the plane $y+z=2$

Plot $\vec{r}(0)$ (as a point)

$\vec{r}\,'(0)$ (as a vector starting at $\vec{r}(0)$)



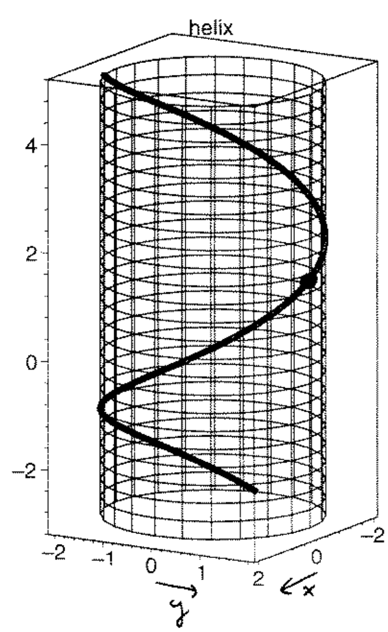
Example 4

Consider the helix

$$\vec{r}(t) = \langle \underbrace{2\cos t, 2\sin t}_{\text{proj to x-y plane is going around circle radius 2}}, \underbrace{t}_{\text{increasing at const rate in z-dir.}} \rangle$$

plot $\vec{r}(\pi/2)$

$\vec{r}\,'(\pi/2)$



Physics: If t = time

$$\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \text{average velocity}$$ net displacement per unit time

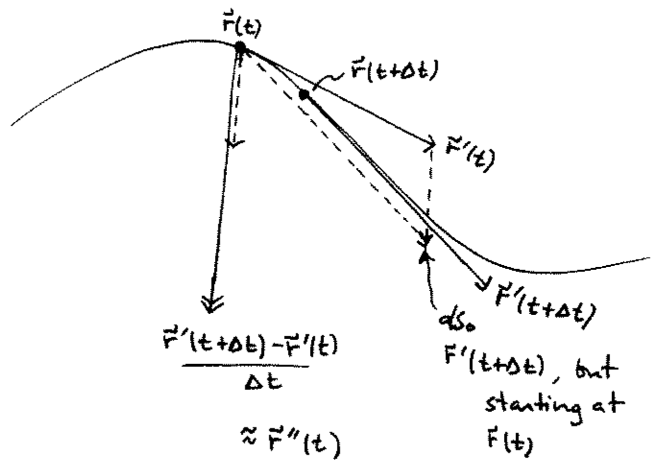
$$\left|\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}\right| = \text{average speed}$$ $\frac{\text{net distance}}{\text{time taken}}$

$\vec{r}\,'(t)$ = velocity $\vec{v}(t)$

$|\vec{r}\,'(t)|$ = speed $v(t)$

$$\boxed{\frac{d\vec{v}}{dt} = \vec{r}\,''(t) = \underset{\vec{a}(t)}{\text{acceleration}}}$$ Measures rate of change of velocity!

Example 5 Compute the acceleration vector for examples 1, 3, 4.
Add to pictures



Curve length: $\vec{r}:[a,b] \to \mathbb{R}^n$

$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

- see math derivation § 6.4 page 295 for plane curve
- makes sense since $\underbrace{\underset{\text{speed}}{|\vec{r}\,'(t)|}\ \underset{\text{time}}{dt}}_{\text{distance traveled in time } dt}$

Example 6 Find the length of one loop of the helix, $\vec{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$
$0 \le t \le 2\pi$
in example 4