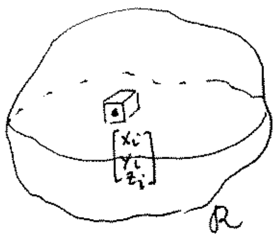
Math 2210-4
Wed 4/6
§ 16.6 triple integrals



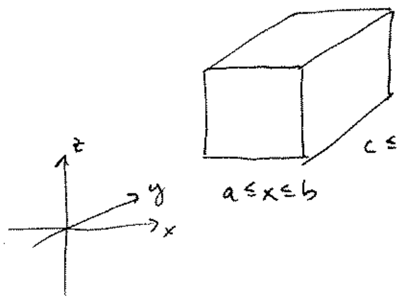

$$\iiint_R f(x,y,z)\,dV \underset{\substack{\text{partition}\\ \text{size}\to 0}}{:= \lim} \sum f(x_i, y_i, z_i)\,\Delta V_i$$

volume (↓ $dV$)

in rectangular coords this could be $(\Delta x_i)(\Delta y_i)(\Delta z_i)$

triple integral will exist for "nice regions" & continuous fcns.

Easiest case is a rectangular prism

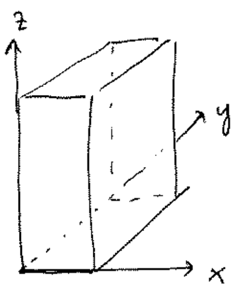



depending on how you chop, there are 6 possible iterated integral expressions for

$$\iiint_R f\,dV$$

<u>Example</u> Consider the "cube"

$0 \le x \le 1$
$0 \le y \le 2$ cm
$0 \le z \le 3$




suppose it has varying density

$$\delta(x,y,z) = (x+1) \quad g/cm^3$$

Find its total mass. One way:

$$\int_0^3 \left( \int_0^2 \left( \int_0^1 (x+1)\,dx \right) dy \right) dz$$

$$\left.\frac{x^2}{2} + x\right]_0^1$$

$$3/2$$

so, ans $= \frac{3}{2} \cdot 2 \cdot 3 = 9$ gm.
(makes sense, since average density is $3/2$ gm/cm$^3$ & volume is 6 cm$^3$).

typical applications of triple integrals:

if $\delta(x,y,z)$ is a (mass) density,

$$m = \iiint_{\mathcal{R}} \delta \, dV \qquad \text{mass.}$$

$$V = \iiint_{\mathcal{R}} 1 \, dV \qquad \text{volume}$$

$$\left.\begin{aligned} \bar{x} &= \frac{1}{m} \iiint_{\mathcal{R}} x \, \delta \, dV \\ \bar{y} &= \frac{1}{m} \iiint_{\mathcal{R}} y \, \delta \, dV \\ \bar{z} &= \frac{1}{m} \iiint_{\mathcal{R}} z \, \delta \, dV \end{aligned}\right\} \text{coords of centroid } \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}$$

$$\left.\begin{aligned} I_x &= \iiint_{\mathcal{R}} (y^2+z^2) \, \delta \, dV \\ I_y &= \iiint_{\mathcal{R}} (x^2+z^2) \, \delta \, dV \\ I_z &= \iiint_{\mathcal{R}} (x^2+y^2) \, \delta \, dV \end{aligned}\right\} \text{moments of inertia.}$$

<u>example</u>. (reason for computing these)

If $\mathcal{R}$ is rotating about $x$-axis with angular velocity

$\omega$ radians/sec.

then velocity of a point $(x,y,z)$ in $\mathcal{R}$ is $\omega \underbrace{\sqrt{y^2+z^2}}_{\text{distance to } x\text{-axis}}$ (as $\mathcal{R}$ is rotating abt $x$-axis)

so kinetic energy

$$KE = \iiint \underbrace{\tfrac{1}{2}}_{\frac{1}{2}} \underbrace{(\delta \, dV)}_{\text{mass}} \underbrace{(\omega^2 (y^2+z^2))}_{(\text{vel})^2}$$

$$= \frac{1}{2} I_x \, \omega^2$$

For general regions, in order to iterate a triple integral, you hope the region is simple above one of the coord. planes

(or you decompose the region into pieces which are.)

Math 2210-4
April 6, 2005
Rectangular prism example

We find the center of mass of the object on page 1 of today's notes, as well as its moments of inertia:

```
> m:=int(int(int(x+1,x=0..1),y=0..2),z=0..3);
    #here is the mass of our object
```

$$m := 9$$

```
> X:=(1/m)*int(int(int(x*(x+1),x=0..1),y=0..2),z=0..3);
  Y:=(1/m)*int(int(int(y*(x+1),x=0..1),y=0..2),z=0..3);
  Z:=(1/m)*int(int(int(z*(x+1),x=0..1),y=0..2),z=0..3);
    #these are the coordinates of the centroid. Do they
    #make sense?
```

$$X := \frac{5}{9}$$

$$Y := 1$$

$$Z := \frac{3}{2}$$

```
> Ix:=int(int(int((y^2+z^2)*(x+1),x=0..1),y=0..2),z=0..3);
  Iy:=int(int(int((x^2+z^2)*(x+1),x=0..1),y=0..2),z=0..3);
  Iz:=int(int(int((y^2+x^2)*(x+1),x=0..1),y=0..2),z=0..3);
    #these are the moments of intertia
```

$$Ix := 39$$

$$Iy := \frac{61}{2}$$

$$Iz := \frac{31}{2}$$

As you can see, if you can set up an integral, the comptuter can compute it for you. (Or approximate its value, if an exact value is unobtainable.)

try one of these by hand:

<u>example</u> (almost an old HW problem)

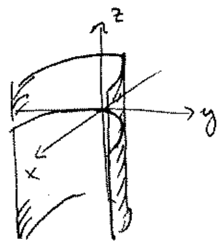
A region $T$ is bounded by

$x = y^2$ (parabolic cylinder)

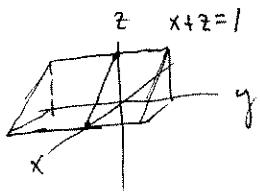
$z = 0$ (xy plane)

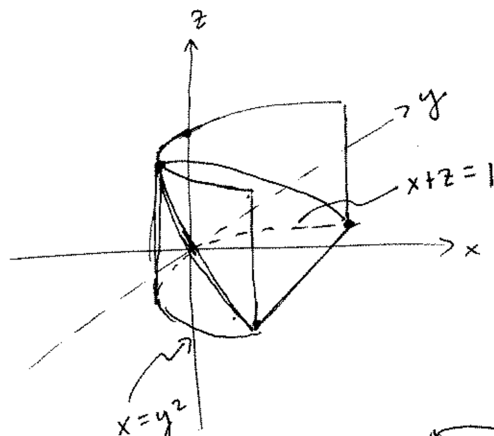
$x + z = 1$ (plane)

Find its volume, and its centroid (assuming constant density).
(center of mass)




$x = y^2$



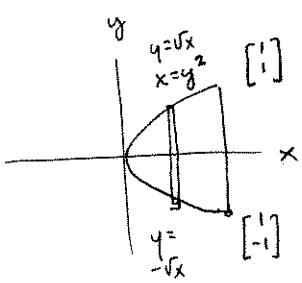




our object

Set up the integrals: Region is simple over xy plane (vertically simple)
also "over" yz plane
<u>and</u> xz plane.

one way

$$V = \int_0^1 \left( \int_{-\sqrt{x}}^{\sqrt{x}} \left( \int_0^{1-x} 1 \, dz \right) dy \right) dx$$




$$\int_{-\sqrt{x}}^{\sqrt{x}} (1-x)\, dy$$

$$\int_0^1 2\sqrt{x}\,(1-x)\, dx$$

$$2x^{1/2} - 2x^{3/2}$$

$$2 \cdot \tfrac{2}{3} x^{3/2} - 2 \cdot \tfrac{2}{5} x^{5/2} \Big]_0^1 = \frac{4}{3} - \frac{4}{5} = 4\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{8}{15}}$$

Try setting up this integral using different iterations!

Example page 4 computations:

```
> V:=int(int(int(1,z=0..1-x),x=y^2..1),y=-1..1);
```

$$V := \frac{8}{15}$$

```
> X:=(1/V)*int(int(int(x,z=0..1-x),x=y^2..1),y=-1..1);
  Y:=(1/V)*int(int(int(y,z=0..1-x),x=y^2..1),y=-1..1);
  Z:=(1/V)*int(int(int(z,z=0..1-x),x=y^2..1),y=-1..1);
    #centroid coordinates
```

$$X := \frac{3}{7}$$

$$Y := 0$$

$$Z := \frac{2}{7}$$

```
>
```