Math 2210-4
Review Sheets
4/27/05

Final exam is comprehensive, 30% of total grade.
You may bring one 4" × 6" card;
scientific calculator only.
I'll provide integral tables.

Review session (go over practice final)
Friday 4/29
LCB 121, 10:30-12:30

Exam is Monday, 10:30-12:30,
in our classroom. LCB 219

Approximate percentages on exam, by chapter:

13-14: 20-25% (vectors & curves)
15: 20-25% (derivative concepts for functions of several variables)
16: 25-30% (multiple integration & applications)
17: 25-30% (vector calculus)

Topics overview: (Also see old review sheets).

Basic tools: each tool is defined algebraically, but has value because of its geometric significance

vectors: addition, scalar multiplication, magnitude, unit vectors
dot product: def, geometric meaning, algebra, projections, components
cross product: def, " " , $\perp$ vectors, area, plane eqtns.
lines, planes, conics, quadric surfaces, cylinders
polar, cylindrical, spherical coords

Differentiation

parametric curves

$\vec{r}(t)$, $\vec{r}'(t)$, $\vec{r}''(t)$: computation and meaning
going from a level curve description (in the plane) to a parametric curve, & vise versa
differentiation rules (sums, products, chain)
$\vec{T}$, $\vec{N}$, κ
$\vec{r}''(t) = v'(t)\vec{T} + \kappa v^2 \vec{N}$; computing and/or measuring ($v = |r'| =$ speed, $\kappa =$ curvature)
$\vec{r}(t+\Delta t) \approx \vec{r}(t) + \vec{r}'(t)\Delta t$ tangent approximation

functions of several variables

$f(\vec{x}+\Delta\vec{x}) \approx f(\vec{x}) + \underbrace{\nabla f(\vec{x})\cdot\Delta\vec{x}}_{\text{"}df\text{"}}$ tangent approximation (= differential approx.)

$D_{\vec{u}} f(\vec{x}) = \lim_{t\to 0} \frac{f(x+t\vec{u}) - f(\vec{x})}{t} = \nabla f(\vec{x})\cdot\vec{u} = |\nabla f(x)|\cos\theta$; calculate & estimate

Chain rule (combine tangent functions for $f(\vec{x})$ and $\vec{r}(t)$ to deduce)

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t))\cdot\vec{r}'(t) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} + f_z\frac{dz}{dt} \quad \text{etc.}$$

$\nabla f(\vec{x}) \perp$ to level curve of f thru $\vec{x}$ (or level surface) (eqtns for tangent planes)

Max-min problems: by finding critical points (perhaps after constraint elimination) or via Lagrange multipliers

Integration

Basic integration

$\int_a^b f(x)\,dx$

double or triple integrals over rectangles or coordinate boxes

more complicated iterated integrals

domain $\leftrightarrow$ setting up iterated integrals

area, volume, mass, center of mass, moments of inertia (memorize or write down formulas!)

Polar coords $dA = r\,dr\,d\theta$

Cylindrical coords $dV = r\,dr\,d\theta\,dz$

Spherical coords $dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

More complicated integrals

ALL based on small scale (tangent) approximation

$$\int_C f(\vec{x})\,ds := \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

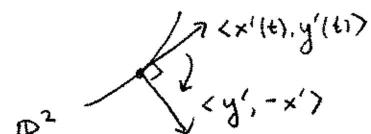
$\vec{x} = \vec{r}(t)$ parametric curve

$d\vec{r} = \vec{r}\,'(t)\,dt$ tangent displacement

$ds = |d\vec{r}| = |\vec{r}\,'(t)|\,dt$ tangent distance (element of arclength)

also written as $\int_C \vec{F}\cdot\vec{T}\,ds$, $\int_C M\,dx + N\,dy + P\,dz$

in $\mathbb{R}^2$, $\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle y', -x'\rangle\,dt$



$d\vec{r} = \vec{T}\,ds = \langle x', y'\rangle\,dt$

$\vec{n}\,ds = \langle y', -x'\rangle\,dt$

$$\iint_S f\,dS = \iint_R f(x,y,g(x,y))\,\underbrace{\sqrt{1+g_x^2+g_y^2}\,dA}_{dS}$$

S ← graph; R ← x-y region; dS for a graph $z = g(x,y)$

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_R \vec{F}(x,y,g(x,y))\cdot\langle -g_x, -g_y, 1\rangle\,dx\,dy$$

S ← graph; R ← x-y region

$dS = \sqrt{1+g_x^2+g_y^2}\,dA$

$\vec{n}\,dS = \langle -g_x, -g_y, 1\rangle\,dA$ (cancellation!)

Fundamental theorems of calculus

$\int_a^b f'(x)\,dx = f(b) - f(a)$

If $\vec{F} = \nabla f$ then $\int_A^B \vec{F}\cdot d\vec{r} = \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)}_{\frac{d}{dt} f(\vec{r}(t))}\,dt = f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A)$

When is $\vec{F} = \nabla f$?

How to find f in this case.

Green's: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S N_x - M_y\,dA$

Stoke's: $\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$

Divergence (Gauss') thm

dim = 2: $\int_C \vec{F}\cdot\vec{n}\,ds = \iint_S \operatorname{div}\vec{F}\,dA$, dim = 3: $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$

Sample final exam

illustrative - not inclusive!

1. a) Sketch the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

   b) Show the parametric curve $\vec{r}(t) = \langle 2\cos t, 3\sin t \rangle$ lies on this ellipse

   c) Compute $\vec{r}(0), \vec{r}\,'(0), \vec{r}\,''(0)$. Add these vectors to your sketch in (a), at appropriate locations

   d) Find the unit vectors $\vec{T}(0)$, $\vec{N}(0)$

   e) Our favorite curve equation decomposes acceleration into tangential and normal components:
   $$\vec{r}\,''(t) = v'(t)\vec{T} + \kappa v^2 \vec{N}$$
   Estimate (with ruler) these tangential and normal components, and compare with the exact values (which could be computed using formulas or with dot product.)

   f) for $f(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$, $\vec{r}(t) = \langle 2\cos t, 3\sin t \rangle$, use the multivariable chain rule to compute
   $$\frac{d}{dt} f(\vec{r}(t)).$$
   Explain why you know the answer will be zero.

2 a) Sketch the surface $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$

   b) Find an equation for its tangent plane at the point $(\sqrt{3}, 0, 2)$

3. Use Green's Theorem with $\vec{F} = \langle -y, x \rangle$, region S the inside of the ellipse in problem 1, and C is boundary curve (parameterized by $\vec{r}(t)$ in problem 1), to prove that the area of the S is 6π.

4. a) Compute $\int_0^4 \int_{y/2}^{\sqrt{y}} 4x \, dx \, dy$

   b) Sketch the region of integration in (a)

   c) Express the integral in 4a) as an iterated integral with reversed order of integration, and compute this integral.

5. a) One of these vector fields is a gradient vector field, and one is not. Identify and explain

(i) $\vec{F} = \langle \sin x + e^x \cos y,\ 3y^2 - e^x \sin y \rangle$

(ii) $\vec{F} = \langle -y, x \rangle$

b) Find a function $f(x,y)$ with $\nabla f = \vec{F}$, for the gradient field in (a)

c) For the gradient field in (a), compute

$$\int_C \vec{F} \cdot d\vec{r} \quad \text{where } C \text{ is any curve from } (0,0) \text{ to } (\pi/2, \pi/2)$$

6. a) If $f(x,y,z) = xy^2/(1+z^2)$, in what (unit) direction is f increasing most rapidly, at the point $(1,1,1)$?

b) Use differentials to approximate $f(1.01, 1.98, 2.03)$.

7. You must build a rectangular shipping crate with volume 60 ft^3. Its top costs \$2/square ft, its sides cost \$1/ft^2, and its bottom costs \$3/ft^2. What dimensions minimize total cost?

8. A uniform wire of density 2 gm/meter is shaped like the semicircle $x^2+y^2=4$, $y \geq 0$. (i.e. its radius is 2 meters)

a) Find the mass of the wire

b) Find its center of mass.

9. Find the volume of the region that lies inside the sphere $x^2+y^2+z^2=9$ but outside the cylinder $x^2+y^2=4$

10. Consider the vector field $\vec{F} = \langle x, y \rangle$, and the square with vertices $(0,0), (2,0), (0,2), (2,2)$.

a) Compute the flux integral $\int_C \vec{F} \cdot \vec{n}\, ds$ around the sides of the square, by identifying $\vec{n}$ and $\vec{F} \cdot \vec{n}$ along each edge.

b) Use the divergence theorem to recompute the value of the flux integral in (a). (Your answers should agree!)