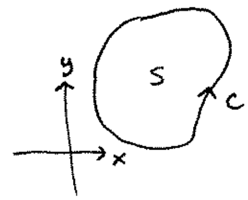
Math 2210-4
Monday 4/25
§ 17.4-17.7

HW problem session
tomorrow (Tuesday)
11:50-12:40 AEB 310
HW due Wed!!!

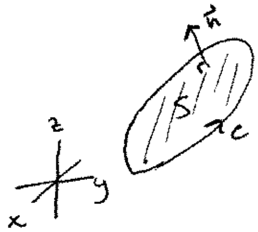
Vector Calculus thms

Green's thm: $\displaystyle\oint_C M\,dx + N\,dy = \iint_S N_x - M_y \, dA$



is a special case of

Stokes thm $\displaystyle\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$



[you can prove Stokes's from Green's
for a piece of surface $z = f(x,y)$,
if you're very good at line integrals & the chain rule]

[you get G from S using $\vec{F} = \langle M, N, 0\rangle$
$\nabla\times\vec{F} = \langle 0, 0, N_x - M_y\rangle$
$\vec{n} = \langle 0, 0, 1\rangle$]

this lets us understand
that curl $\vec{F}$ is measuring "circulation"

Example $\vec{F} = \langle 0, -z, y\rangle$

compute curl $\vec{F}$, and interpret
using Stoke's thm:

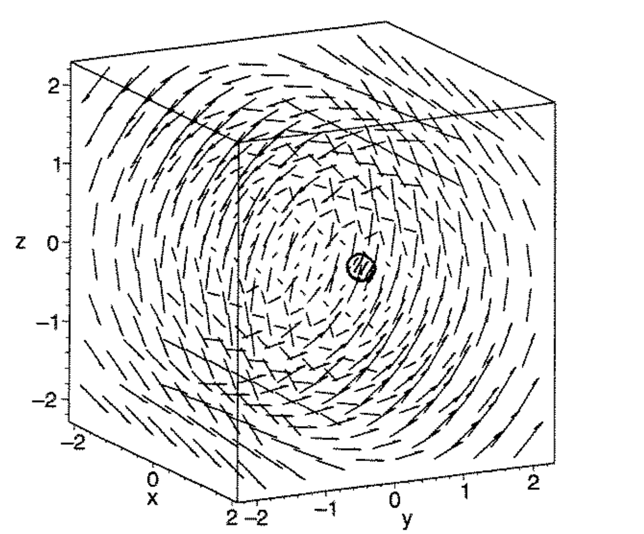
[for tiny disks, $\oint \vec{F}\cdot\vec{T}\,ds$ will
be largest if the disk has
normal // to curl $\vec{F}$ –
this is an infinitesimal
axis of rotation]

vector fields, div, curl:

```
> with(plots):
> M:=(x,y,z)->0;
  N:=(x,y,z)->-z;
  P:=(x,y,z)->y;
```

$$M := 0$$
$$N := (x, y, z) \to -z$$
$$P := (x, y, z) \to y$$

```
> fieldplot3d([M(x,y,z),N(x,y,z),P(x,y,z)],x=-2..2,y=-2..2,
  z=-2..2,color=black,axes=boxed);
```



If we project $\perp$ to the rotation axis:

vector fields, div, curl:

```
> with(plots):
> M:=(x,y,z)->0;
  N:=(x,y,z)->-z;
  P:=(x,y,z)->y;
```

$$M := 0$$

$$N := (x, y, z) \rightarrow -z$$

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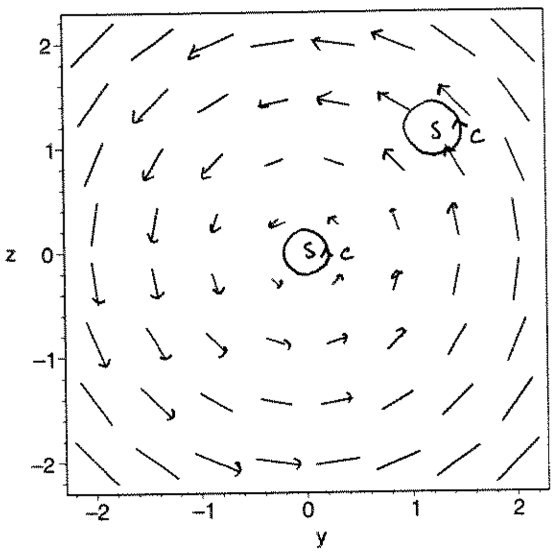
```
> fieldplot3d([M(x,y,z),N(x,y,z),P(x,y,z)],x=-2..2,y=-2..2,
  z=-2..2,color=black,axes=boxed);
```

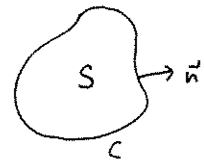
$\operatorname{curl}\vec{F} = \langle 2,0,0\rangle$

$\operatorname{curl}\vec{F}\cdot\vec{n}$
$= 2$

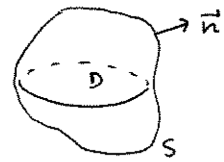


Vector Calc Thms cont'd.

$n=2$ div thm $\displaystyle\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_S \operatorname{div}\vec{F}\,dA$

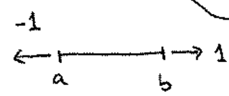


$n=3$ div thm $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_D \operatorname{div}\vec{F}\,dV$



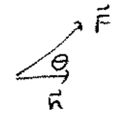
$n=1$ div thm! $\displaystyle F(b)-F(a) = \int_a^b F'(x)\,dx$

FTC

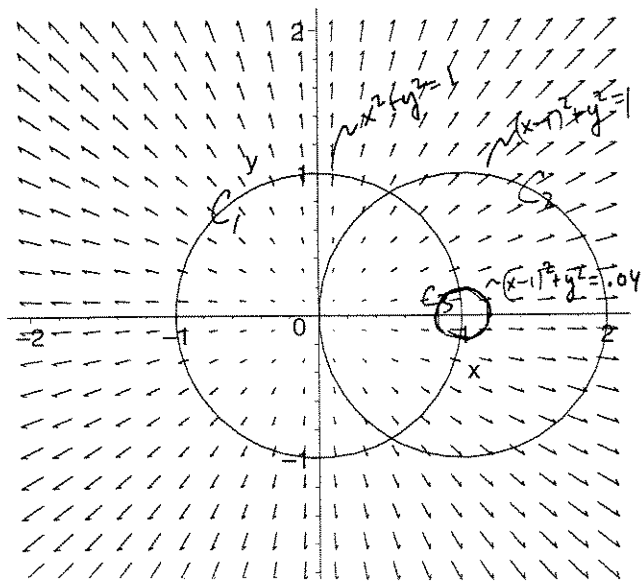


Notice – this theorem tells you what "divergence" is measuring!

$\vec{F}\cdot\vec{n} = |\vec{F}|\cdot 1\,\cos\theta$

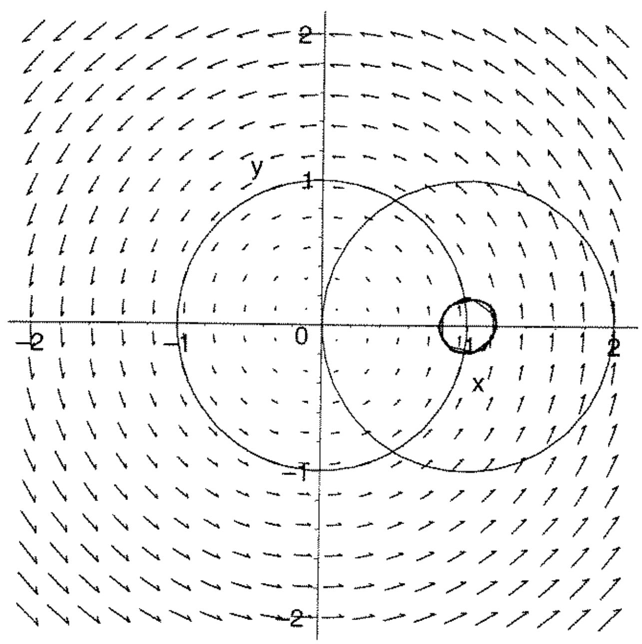


F=<x,y>



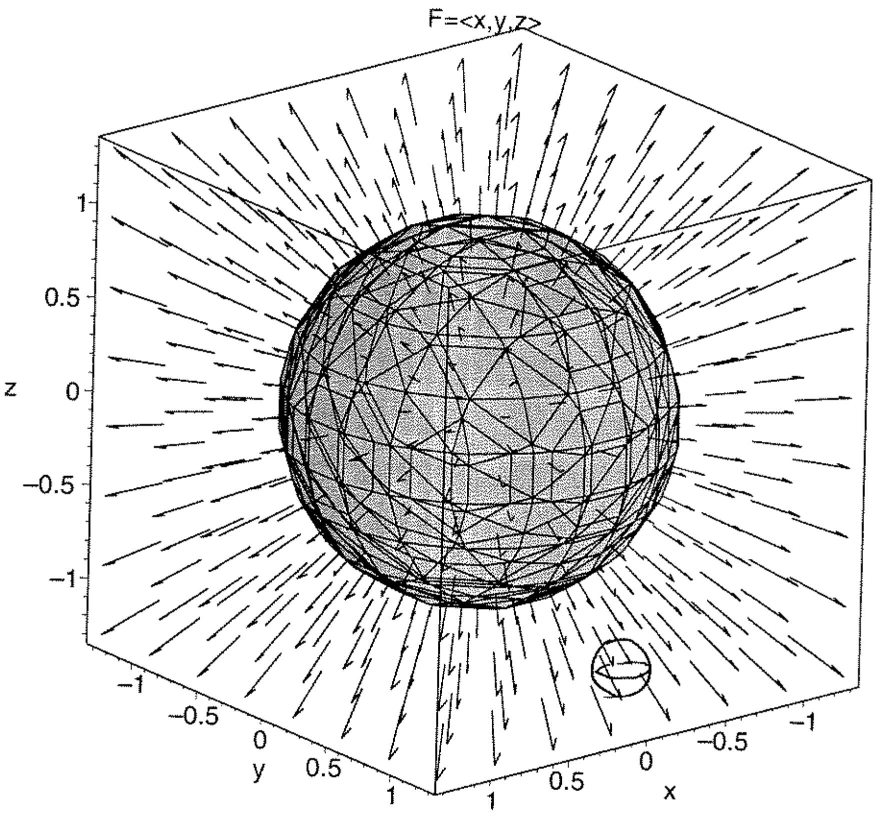
$\operatorname{div}\vec{F}=2$

G=<-y,x>



$\operatorname{div}\vec{G}=0$

$n=3$ div thm



$\operatorname{div} \vec{F} = 3$

All of these theorems are really just the F.T.C. in disguise

$n=2$ div $\Rightarrow$ $n=2$ Green (we went the other direction in class notes, but you can reverse it.)

$\Rightarrow$ Stoke's (see page 1)

FTC $\Rightarrow$ each dimension's div thm:

$n=3$ (FTC)

$$\iiint_D \frac{\partial f}{\partial x}\, dV = \iint_S f\, \hat{\imath}\cdot\vec{n}\, dS$$

$$\iiint_D \frac{\partial f}{\partial y}\, dV = \iint_S f\, \hat{\jmath}\cdot\vec{n}\, dS$$

$$* \quad \iiint_D \frac{\partial f}{\partial z}\, dV = \iint_S f\, \hat{k}\cdot\vec{n}\, dS$$

$n=2$

$$\iint_S \frac{\partial f}{\partial x}\, dA = \int_C f\, \hat{\imath}\cdot\vec{n}\, ds$$

$$\iint_S \frac{\partial f}{\partial y}\, dA = \int_C f\, \hat{\jmath}\cdot\vec{n}\, ds$$

$n=1$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

so,

$$\iint_S M_x + N_y = \int_C M\,\hat{\imath}\cdot\vec{n} + N\,\hat{\jmath}\cdot\vec{n}\, ds$$

$$= \int_C \langle M, N\rangle\cdot\vec{n}\, ds \quad \boxed{n=2 \text{ div thm}}$$

so,

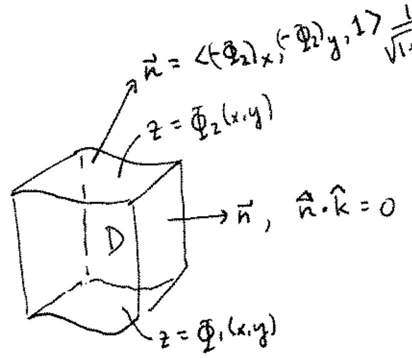
$$\iiint_D M_x + N_y + P_z\, dV$$

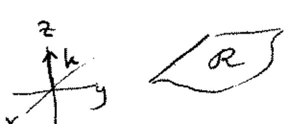
$$= \iint_S M\,\hat{\imath}\cdot\vec{n} + N\,\hat{\jmath}\cdot\vec{n} + P\,\hat{k}\cdot\vec{n}\, dS$$

$$\boxed{n=3 \text{ div thm}} \quad = \iint_S (M\hat{\imath} + N\hat{\jmath} + P\hat{k})\cdot\vec{n}\, dS = \iint_S \vec{F}\cdot\vec{n}\, dS$$

check

for a z-simple region





decompose a given region into z-simple ones and add up the pieces ∎

$$\iiint_D f_z\, dV = \iint_{\mathcal{R}} \int_{\Phi_1(x,y)}^{\Phi_2(x,y)} \frac{\partial f}{\partial z}\, dz\, dA$$

$$= \iint_{\mathcal{R}} f(x,y,\Phi_2(x,y))\, dA - \iint_{\mathcal{R}} f(x,y,\Phi_1(x,y))\, dA$$

$$= \iint_S f\, \hat{k}\cdot\vec{n}\, dS$$