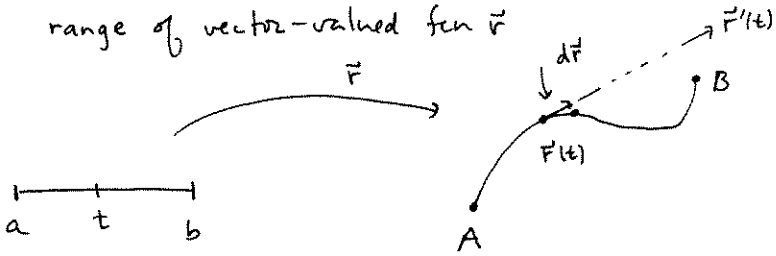
Math 2210-4
Fri 4/22

§17.4 - 17.7

HW for Wed 4/27
§17.4 9, 14
§17.5 1, 3 9 11
§17.6 1 5 15
§17.7 3 9 19

Curve $\vec{x} = \vec{r}(t)$,
range of vector-valued fcn $\vec{r}$



substitutions: $\boxed{\begin{aligned} d\vec{r} &= \vec{r}'(t)\,dt \\ ds = |d\vec{r}| &= |\vec{r}'(t)|\,dt \end{aligned}}$ infinitesimal (tangent) displacement vector
element of arclength.

Thus (A) $\displaystyle\int_C f(\vec{x})\,ds \;:=\; \int_a^b f(\vec{r}(t))\,\underbrace{|\vec{r}'(t)|\,dt}_{ds}$ curve integral of a scalar fcn (e.g. for mass, ctr mass etc.).

(B) $\displaystyle\int_C \vec{F}\cdot d\vec{r} \;:=\; \int_a^b \vec{F}(\vec{r}(t))\cdot \underbrace{\vec{r}'(t)\,dt}_{d\vec{r}}$ line integral: work done by a force field.

$\Vert$

$\displaystyle\int_C \vec{F}\cdot\vec{T}\,ds$

notice $d\vec{r} = \vec{r}'(t)\,dt = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}\,|\vec{r}'(t)|\,dt = \vec{T}\,ds$

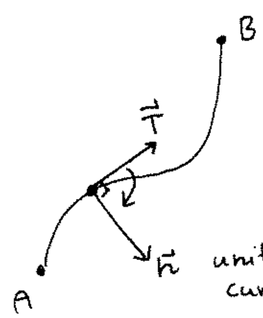
$\uparrow$ unit tangent vector.

NEW TODAY!!

(C) $\displaystyle\int_C \vec{F}\cdot\vec{n}\,ds \;:=\; \int_a^b \vec{F}(\vec{r}(t))\cdot \langle y'(t), -x'(t)\rangle\,dt$ $\boxed{\text{flux integral}}$

$$\vec{T} = \frac{\langle x'(t), y'(t)\rangle}{|\vec{r}'(t)|}$$

so $\vec{n} = \dfrac{\langle y'(t), -x'(t)\rangle}{|\vec{r}'(t)|}$; so $\vec{n}\,ds = \langle y'(t), -x'(t)\rangle\,dt$



unit normal to curve (rotate $\vec{T}$ $-\pi/2$ radians)

Example 1

Let $C = \{(x,y) \mid x^2+y^2=4\}$

$\vec{F} = \langle x, y \rangle$

Compute the flux integral

$$\oint_C \vec{F}\cdot\vec{n}\,ds$$

(a) by parameterizing the curve

(b) by being smart

$n=2$ divergence theorem

If C is the boundary of a bounded region S, then, for $\vec{n}$ = unit exterior normal,

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds = \iint_S \operatorname{div}\vec{F}\,dA}$$

proof:

$$\int_C \langle M,N\rangle \cdot \vec{n}\,ds = \int_a^b \langle M(\vec{r}(t)), N(\vec{r}(t))\rangle \cdot \langle y'(t), -x'(t)\rangle\,dt$$

$$= \int_C -N\,dx + M\,dy \quad !$$

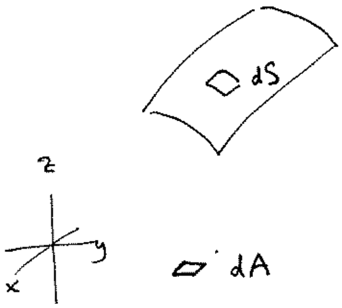
$$= \iint_S M_x - (-N)_y\,dA \quad \text{by Green's Thm!}$$

$$= \iint_S \operatorname{div}\vec{F}\,dA \quad \blacksquare$$

Example 2: Check Example 1 with div thm.

Surface integrals

Recall, for a graph $z = f(x,y)$



$$\boxed{dS = \sqrt{1+f_x^2+f_y^2}\; dA}$$

↑ area expansion factor due to inclination of graph.

Also, recall eqtn of tangent plane was

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

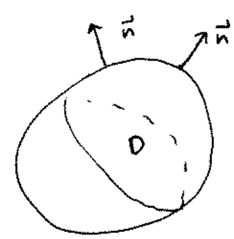
so $\langle -f_x, -f_y, 1\rangle$ is $\perp$ to tangent plane.

so unit upper normal is

$$\boxed{\vec{n} = \frac{1}{\sqrt{1+f_x^2+f_y^2}} \langle -f_x, -f_y, 1\rangle}$$

thus

$$\boxed{\vec{n}\, dS = \langle -f_x, -f_y, 1\rangle\, dA}$$



$n=3$ divergence theorem

Let D be a bounded region in $\mathbb{R}^3$ with boundary surface ∂D. Let $\vec{n}$ be the unit exterior normal to ∂D. Then

$$\boxed{\iint_{\partial D} \vec{F}\cdot\vec{n}\, dS = \iiint_D \operatorname{div}\vec{F}\, dV}$$

← generalizes $n=2$ divergence theorem

Stoke's theorem:

Let S be a surface in $\mathbb{R}^3$, with boundary curve C. Let $\vec{n}$ be the unit normal on $\vec{n}$, oriented so that $\vec{T}\times\vec{n}$ points away from S, along C. Then

$$\boxed{\iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\, dS = \oint_C \vec{F}\cdot\vec{T}\, ds}$$

↙ generalizes $n=2$ Green's theorem

Why? → Monday.

But now you can attempt all HW.

Example $\vec{F} = \langle x, y, z \rangle$

$D = \{ x^2 + y^2 + z^2 \leq 4 \}$

Verify the divergence theorem in this case!