Math 2210-4
Wed 4/20
17.4 Green's thm → Stoke's Thm
Divergence Thm.

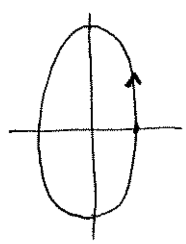
- Monday's notes contain
  - the statement of Green's Thm
  - its proof
  - an example.

After going through that material, try using Green's thm to evaluate Example:

$$\oint_C (x^2+2y+xy)\,dx + (4x-3y^2)\,dy \qquad = \iint_S (N_x - M_y)\,dA$$

where C is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

$\vec{r}(t) = \langle 2\cos t, 3\sin t\rangle$
$\vec{r}'(t) = \langle -2\sin t, 3\cos t\rangle$



Of course, Maple could compute the line integral directly:

```
> Int((4*cos(t)^2+6*sin(t)+6*cos(t)*sin(t))*(-2*sin(t)
  )+(4*2*cos(t)-3*(9*sin(t)^2))*3*cos(t),t=0..2*Pi);
```

$$\int_0^{2\pi} -2\,(4\cos(t)^2+6\sin(t)+6\cos(t)\sin(t))\sin(t)+3\,(8\cos(t)-27\sin(t)^2)\cos(t)\,dt$$

```
> int((4*cos(t)^2+6*sin(t)+6*cos(t)*sin(t))*(-2*sin(t)
  )+(4*2*cos(t)-3*(9*sin(t)^2))*3*cos(t),t=0..2*Pi);
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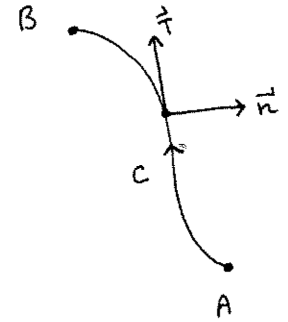
$$12\pi$$

Flux integrals

$\vec F = \langle M, N\rangle$ vector field.



flux integral :

$$\int_C \vec F \cdot \vec n \, ds$$

$$= \int_a^b \vec F(\vec r(t)) \cdot \langle y'(t), -x'(t)\rangle \frac{\cancel{|\vec r'(t)|}}{\cancel{|\vec r'(t)|}} dt$$

curve $\vec x = \vec r(t)$, $a \le t \le b$

$\vec T$ = unit tangent vector;

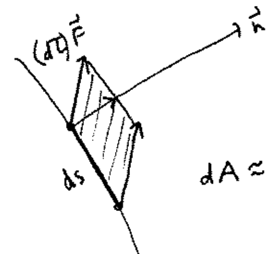
$$\vec T = \frac{\vec r'(t)}{|\vec r'(t)|} = \frac{\langle x', y'\rangle}{|\vec r'|}$$

$\vec n$ = unit normal (oriented as shown)

$$= \frac{\langle y', -x'\rangle}{|\vec r'|}$$

called "flux" because one interpretation considers $\vec F$ as a velocity vector field of moving fluid film (2-d), and $\int_C \vec F \cdot \vec n \, ds$ measures how fast the fluid is moving across C (area/time):

magnify: τ = time, $d\tau$ an increment



$$dA \approx ds\,(d\tau\, \vec F \cdot \vec n)$$

$$A(d\tau) - A(0) \approx d\tau \int_C \vec F \cdot \vec n \, ds$$

$$\frac{dA}{d\tau} = \int_C \vec F \cdot \vec n \, ds.$$

Green's thm lets us evaluate flux integrals around closed loops :

$n=2$

Div thm :

$$\iint_S \operatorname{div} \vec F \, dA = \oint_C \vec F \cdot \vec n \, ds$$

proof (see above)

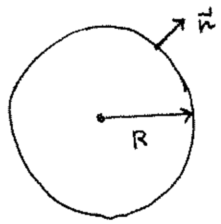
$$\oint_C \vec F \cdot \vec n \, ds = \int_a^b \langle M, N\rangle \cdot \langle y', -x'\rangle \, dt$$

$$= \int_a^b \langle -N, M\rangle \cdot \langle x', y'\rangle \, dt$$

$$= \oint_C -N\,dx + M\,dy = \iint_S M_x - -N_y \, dA = \iint_S M_x + N_y \, dA = \iint_S \operatorname{div} \vec F \, dA$$

example

$\vec{F} = \langle x, y \rangle$, S is the disk of radius R



$$\int_C \vec{F} \cdot \vec{n} \, ds =$$

$$\iint_S \vec{\nabla} \cdot \vec{F} \, dA =$$

Physics application

$\vec{F}$ = force field.

$\mathcal{C}$ connects A to B

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x} \;:=\; \underset{W}{\text{work done}} \text{ by } \vec{F} \text{ to move object from A to B along } \mathcal{C}$$

$$PE := -\int_C \vec{F}\cdot d\vec{x} \quad = \text{work done by object} \qquad \text{(Potential energy)}$$

$$KE := \tfrac{1}{2}mv^2 \quad , \quad v = |\vec{r}\,'(t)| \quad , \quad \text{Kinetic energy.}$$

If $\vec{F}$ is a gradient vector field, it is called <u>conservative</u>, because
$\vec{F} = \nabla f$

$$\text{Newton } (\vec{F} = m\vec{r}\,''(t)) \implies PE + KE \equiv \text{constant for partical motion.}$$

<u>proof</u>

$$PE := -\int_{\vec{r}(0)}^{\vec{r}(t)} \vec{F}\cdot d\vec{x} \;=\; f(\vec{r}(0)) - f(\vec{r}(t))$$

$$KE = \tfrac{1}{2}\, m\, \vec{r}\,'(t)\cdot\vec{r}\,'(t)$$

$$\frac{d}{dt}(KE+PE) = \frac{d}{dt}\left(\tfrac{1}{2}m\vec{r}\,'\cdot\vec{r}\,' + f(\vec{r}(0)) - f(\vec{r}(t))\right)$$

$$= \underset{\text{(product rule)}}{m\vec{r}\,'\cdot\vec{r}\,''} - \underset{\text{(chain rule)}}{\nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)}$$

$$= \vec{r}\,'\cdot\Big[\underbrace{m\vec{r}\,'' - \vec{F}(\vec{r}(t))}_{=0 \text{ by Newton!}}\Big]$$

<u>examples</u>

<u>uniform grav. field</u>

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix} = \nabla(-mgz)$$

$$PE = mgz \quad (+\text{const})$$

<u>inverse square law</u>

$$\vec{F} = \left(-\frac{c}{|\vec{r}|^3}\right)\vec{r}$$

$$f = \frac{c}{|\vec{r}|}$$

$$PE = -\frac{c}{|\vec{r}|} \qquad (+\text{ const.})$$