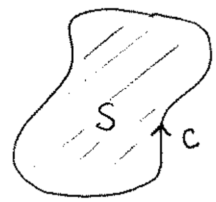
Math 2210-4
Monday 4/18
17.3-17.4 (more 17.4 Wed.)

We will finish Friday's notes on conservative vector fields.

At one point we will use <u>Green's Theorem</u> (§17.4)

$$\boxed{\int_C M\,dx + N\,dy = \iint_S (N_x - M_y)\,dA}$$

scalar curl of $\langle M, N \rangle$

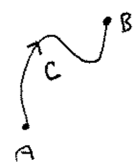


S is a region, C is its boundary oriented counter-clockwise.

<u>Friday page</u> $3\frac{1}{2}$

We already showed that if $\vec{F} = \nabla f$

Then $\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A)$



so the line integral does not depend on the particular path from A to B.

On page $3\frac{1}{2}$ we show that if line integrals for $\vec{F}$ are path independent, then we can use them to construct f with $\nabla f = \vec{F}$

In fact, we show (via Green's thm on a rectangle), that if $N_x = M_y$, such an f exists!
(without having to show path indepence for all paths.)

- do pages $3\frac{1}{2}$-4 Fri.

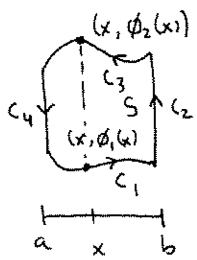
Proof of Green's Thm

(1) $\displaystyle\iint_S M_y\,dA = \oint_{\partial S} -M\,dx$

(2) $\displaystyle\iint_S N_x\,dA = \oint_{\partial S} N\,dy$

(3): (2)-(1) $\Rightarrow$ $\displaystyle\iint_S N_x - M_y\,dA = \int_{\partial S} M\,dx + N\,dy$ ∎ (After (1)(2)).

(1) If S is y-simple:



$$\iint_S M_y\,dA = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} M_y\,dy \right) dx$$

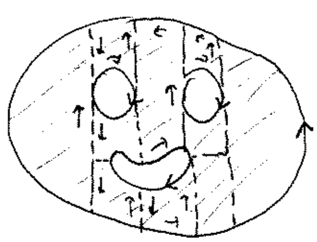
$$= \int_a^b M(x,y) \Big]_{\phi_1(x)}^{\phi_2(x)} dx$$

$$= \int_a^b M(x, \phi_2(x))\,dx - \int_a^b M(x, \phi_1(x))\,dx$$

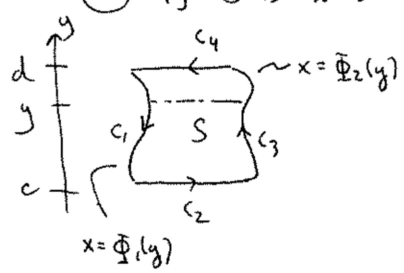
$$= -\int_{C_3} M\,dx - \int_{C_1} M\,dx$$

$$= -\oint_C M(x,y)\,dx$$ because $dx \equiv 0$ on C_4 & C_2!

For general S, decompose into vertically-simple subdomains and add up Green's identity over all of them. Line integrals on inside cancel out, giving result!



(2) If S is x-simple



$$\iint_S N_x\,dA = \int_c^d \int_{\Phi_1(y)}^{\Phi_2(y)} N_x(x,y)\,dx\;dy$$

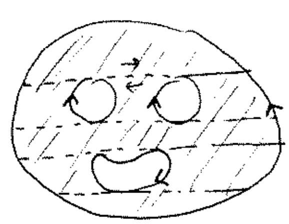
$$= \int_c^d N(x,y) \Big]_{x=\Phi_1(y)}^{x=\Phi_2(y)} dy$$

$$= \int_c^d N(\Phi_2(y), y)\,dy - \int_c^d N(\Phi_1(y), y)\,dy$$

$$= \int_{C_3} N(x,y)\,dy + \int_{C_1} N(x,y)\,dy$$

$$= \oint_C N(x,y)\,dy$$

General S:

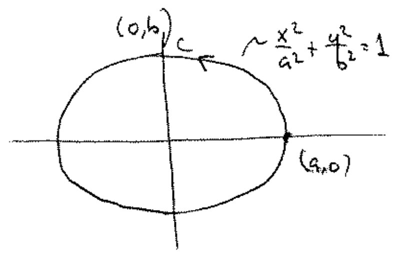


∎

Example : Find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ using Green's Theorem.

Hint : $\vec{r}(t) = \langle \underset{x}{\underset{\parallel}{a\cos t}}, \underset{y}{\underset{\parallel}{b\sin t}} \rangle$

and use $\langle M, N \rangle = \langle -y, x \rangle$



Physics application

$\vec{F}$ = force field.

$\mathcal{C}$ connects A to B

$$\int_{\mathcal{C}} \vec{F}\cdot d\vec{x} \;:=\; \underset{W}{\text{work}} \text{ done by } \vec{F} \text{ to move object from A to B along } \mathcal{C}$$

$$PE := -\int_C \vec{F}\cdot d\vec{x} \quad = \text{work done by object} \qquad \text{(Potential energy)}$$

$$KE := \tfrac{1}{2}mv^2 \quad , \quad v = |\vec{r}\,'(t)| \quad , \quad \text{Kinetic energy.}$$

If $\vec{F}$ is a gradient vector field, it is called <u>conservative</u>, because
$\vec{F} = \nabla f$

$$\text{Newton } (\vec{F} = m\vec{r}\,''(t)) \implies PE + KE \equiv \text{constant for partical motion.}$$

<u>proof</u>

$$PE := -\int_{\vec{r}(0)}^{\vec{r}(t)} \vec{F}\cdot d\vec{x} = f(\vec{r}(0)) - f(\vec{r}(t))$$

$$KE = \tfrac{1}{2}\, m\, \vec{r}\,'(t)\cdot\vec{r}\,'(t)$$

$$\begin{aligned}
\frac{d}{dt}(KE+PE) &= \frac{d}{dt}\left(\tfrac{1}{2}m\vec{r}\,'\cdot\vec{r}\,' + f(\vec{r}(0)) - f(\vec{r}(t))\right) \\
&= \underset{\text{(product rule)}}{m\vec{r}\,'\cdot\vec{r}\,''} - \underset{\text{(chain rule)}}{\nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)} \\
&= \vec{r}\,'\cdot\Big[\underbrace{m\vec{r}\,'' - \vec{F}(\vec{r}(t))}_{=0 \text{ by Newton!}}\Big]
\end{aligned}$$

<u>examples</u>

<u>uniform grav. field</u>

$$\vec{F} = \begin{bmatrix} 0 \\ 0 \\ -gm \end{bmatrix} = \nabla(-\cancel{gm}-mgz)$$

$$PE = mgz \quad (+\text{const})$$

<u>inverse square law</u>

$$\vec{F} = \left(-\frac{\cancel{m}c}{|\vec{r}|^3}\right)\vec{r}$$

$$f = \cancel{m}\frac{c}{|\vec{r}|}$$

$$PE = -\frac{c}{|\vec{r}|} \quad (+\text{ const.})$$