Math 2210-4
Fri 4/15
§17.3: When are line integrals path independent?

HW for Fri 4/22
17.3: 1, 7, 9, 11, 13, 21, 22 — in 13, do the computation both ways: via a path and with a potential fcn.
17.4: 1, 6, 7, 9, 14, 16, 19
17.5: 1, 3, 9, 11, 14, 17

§17.5 might get postponed.

Question:

If C is a curve connecting A to B
For what vector fields $\vec{F}$ does the line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

only depend on the initial pt A and terminal point B,
but not on the particular path C?
In this case, the line integrals for $\vec{F}$ are called path independent

Answer: Line integrals for $\vec{F}$ are path independent if and only if, $\vec{F}$ is a gradient vector field,

$$\vec{F} = \nabla f$$

½
Why: Half the answer is our chapter 15 buddy: the chain rule!!!
If $\vec{F} = \nabla f$, then $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

(since $\int_C \nabla f \cdot d\vec{r} = \int_a^b \underbrace{\nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)}_{\frac{d}{dt} f(\vec{r}(t)) \text{ by chain rule}}\,dt$, where $\vec{r}:[a,b]\to \mathbb{R}^n$ parameterizes C, $\vec{r}(a)=A$, $\vec{r}(b)=B$

$= f(\vec{r}(t))\Big]_a^b$ by 1210 F.T.C.

$= f(\vec{r}(b)) - f(\vec{r}(a))$

$= f(B) - f(A)$.

Example 1

Let $\vec{F} = \langle yz^2, xz^2, 2xyz \rangle$

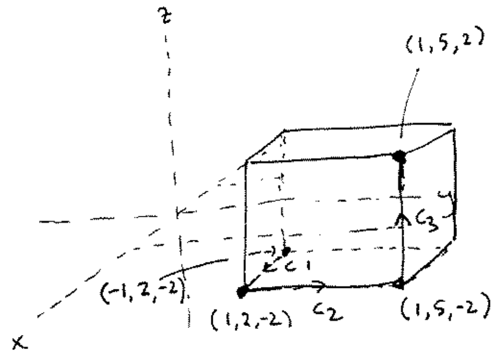
$A = (-1, 2, -2)$

$B = (1, 5, 2)$

1a) Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_C yz^2\,dx + xz^2\,dy + 2xyz\,dz$

by following coord curves:

We should get 28



C_1: $-1 \le x \le 1$, $y = 2$, $z = -2$

C_2: $2 \le y \le 5$, $x = y$, $z = -2$

C_3: $-2 \le z \le 2$, $x = 1$, $y = 5$

Example 1 cont'd

1b) Compute $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ where C is a straight segment connecting $(-1, 2, -2)$ to $(1, 5, 2)$

verify that this line integral also yields 28.

$$\vec{r}(t) = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \qquad 0 \le t \le 1$$

$$\int_C \vec{F}\cdot d\vec{x} = \int_0^1 \langle \underbrace{(2+3t)}_{y}\underbrace{(-2+4t)^2}_{z^2},\ \underbrace{(-1+2t)}_{x}\underbrace{(-2+4t)^2}_{z^2},\ 2\underbrace{(-1+2t)}_{x}\underbrace{(2+3t)}_{y}\underbrace{(-2+4t)}_{z} \rangle \cdot \underbrace{\langle 2, 3, 4 \rangle}_{\vec{r}'(t)}\,dt$$

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> x:=-1+2*t:
  y:=2+3*t:
  z:=-2+4*t:
  int(2*y*z^2+3*x*z^2+4*2*x*y*z,t=0..1);
                                        28
>
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Example cont'd.

1c) Notice $\vec{F} = \nabla f$

where $f(x,y,z) = xyz^2$

Thus $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

where $B = (1,5,2)$

$A = (-1,2,-2)$.

Do you get 28?

Is this way easier?

<u>other ½ why</u>:

If integrals are path independent, then you can use them to find (define) f, with $\nabla f = \vec{F}$.

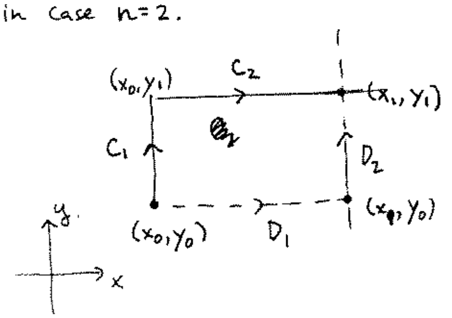
Define $f(\vec{x}) = \displaystyle\int_C \vec{F}\cdot d\vec{r}$, where C connects a fixed base point $\vec{x}_0$ to the varying point $\vec{x}$.

(C is arbitrary since we <u>assume</u> $\vec{F}$ yields path independent line integrals

Then $\nabla f = \vec{F}$

proof: in case $n=2$.

$\vec{F} = \langle M, N\rangle$



thus

$$f(x_1,y_1) = \int_{C_1} N\,dy + \int_{x_0}^{x} M(t,y_1)\,dt$$

so $\dfrac{\partial f}{\partial x}(x_1,y_1) = 0 + M(x_1,y_1)$ (1210 FTC)

Also, $\displaystyle f(x_1,y) = \int_{D_1} M\,dx + \int_{y_0}^{y} N(x_1,t)\,dt$

so $\dfrac{\partial f}{\partial y}(x_1,y_1) = N(x_1,y_1)$ ∎

Test for path independence

If $\vec{F}$ is differentiable on a rectangular domain
then $\vec{F}$ yields path independent line integrals
exactly when $\text{curl}\,\vec{F} \equiv \vec{0}$

$n=2$: $\vec{F} = \langle M, N \rangle$ $\quad N_x = M_y \quad (= f_{xy} = f_{yx})$ ← follows from Green's thm § 17.4

$n=3$ $\quad \vec{F} = \langle M, N, P \rangle$, $\quad \vec{\nabla} \times \vec{F} = \langle 0, 0, 0 \rangle$. ← from Stoke's thm, § 17.7

<u>example 2</u> : Show $\vec{F} = \langle 2x+y, x^2-y^2 \rangle$ is <u>not</u> a gradient field.

<u>example 3</u> : Show $\vec{F} = \langle 6xy - y^3, 4y + 3x^2 - 3xy^2 \rangle$ <u>is</u> a gradient field, and find f so that $\nabla f = \vec{F}$

<u>method 1</u> : $$f(x,y) = \int_{(0,0)}^{(x,y)} \vec{F} \cdot d\vec{r}$$

<u>method 2</u> : "antidifferentiation".