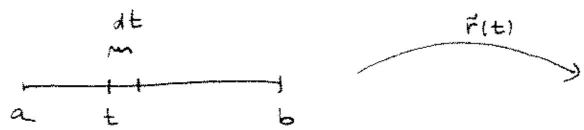
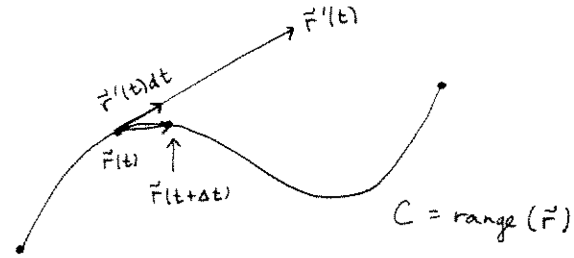
Math 2210-4
Wed 13 April

§17.2 <u>curve integrals</u> and <u>line integrals</u>





$$\Delta \vec r = \vec r(t+dt) - \vec r(t) \approx d\vec r = \vec r\,'(t)\,dt$$

$$\Delta s = |\vec r(t+dt) - \vec r(t)| \approx ds = |d\vec r| = |\vec r\,'(t)|\,dt$$

ds is called the "element of arclength"

In §13.5, 14.5 (and earlier in 1210, §6.4) you computed curve length:

$$L = \int ds = \int_a^b |\vec r\,'(t)|\,dt$$

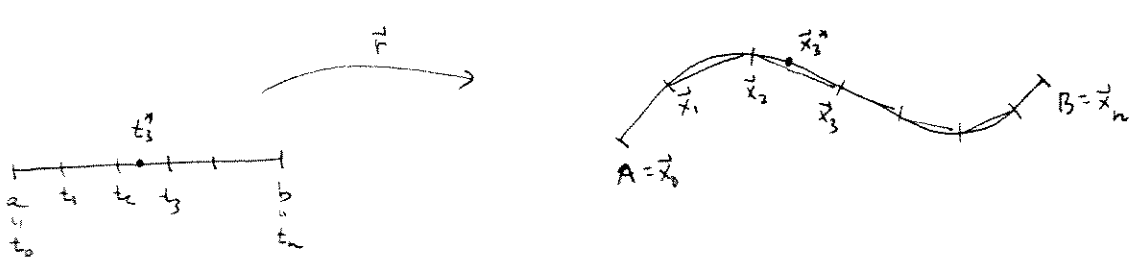
<u>Curve integrals</u> (of scalar functions)

if C is a curve in $\mathbb{R}^n$, the range of a parametric curve $\vec r(t)$ and f is a scalar function from C to $\mathbb{R}$,

$$\int_C f(x,y,z)\,ds := \int_a^b f(\vec r(t))\,\underbrace{|\vec r\,'(t)|\,dt}_{ds}$$

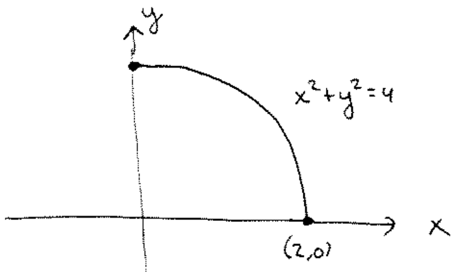
<u>Note</u>: This integral is really a property of C, and not of the particular $\vec r(t)$ parameterizing it; this is because

$$\int_C f(\vec x)\,ds = \lim_{\|P\|\to 0} \sum_i f(x_i^*)\,\underbrace{|\vec x_i - \vec x_{i-1}|}_{\Delta s_i}$$



If $\vec x_i = \vec r(t_i)$
then the Riemann sum above is $\approx \sum_i f(\vec r(t_i^*))\,\underbrace{|\vec r(t_i) - \vec r(t_{i-1})|}_{\approx\ |\vec r\,'(t_{i-1})|\,\Delta t_i}$

Example Let C be the radius 2 quarter-circle $x^2+y^2=4$
$x \geq 0, y \geq 0$



Suppose there is a wire in the shape of C, of density $\frac{xy}{2}$ (e.g. grams/meter)
mass/length.

Find mass $m = \int_C \underbrace{\delta\, ds}_{dm}$: Step 1: Find a good way to parameterize C!

center of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{m} \int_C x\, \delta\, ds$$

$$\bar{y} = \frac{1}{m} \int_C y\, \delta\, ds$$

moments of intertia I_x, I_y, I_z

$$I_x = \int_C y^2\, \delta\, ds$$

$$I_y =$$

$$I_z =$$

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> mass:=int(4*cos(t)*sin(t),t=0..Pi/2);
  xbar:=int(8*cos(t)^2*sin(t),t=0..Pi/2)/mass;
  ybar:=int(8*cos(t)*sin(t)^2,t=0..Pi/2)/mass;
  Ix:=int(4*sin(t)^2*2*cos(t)*2*sin(t),t=0..Pi/2);
  Iy:=int(4*cos(t)^2*2*cos(t)*2*sin(t),t=0..Pi/2);
  Iz:=Ix+Iy;
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$$mass := 2$$

$$xbar := \frac{4}{3}$$

$$ybar := \frac{4}{3}$$

$$Ix := 4$$

$$Iy := 4$$

$$Iz := 8$$

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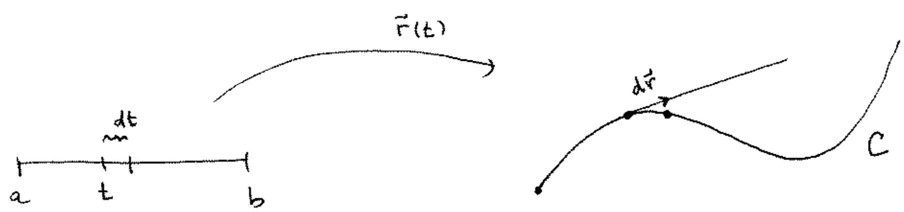
Line integrals : A special type of curve integral use to compute work done by a force vector field. $\vec{F}(x,y,z)$

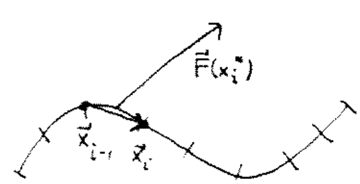


$$\Delta\vec{r} \approx d\vec{r} = \vec{r}\,'(t)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} \quad := \int_a^b F(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

← Really a property of the curve, and the direction you traverse it, since it's a limit of Riemann sums

$$\sum \vec{F}(\vec{x}_i^*)\cdot(\vec{x}_i - \vec{x}_{i-1})$$



If we write

$\vec{F} = \langle M, N, P\rangle$

$d\vec{r} = \langle dx, dy, dz\rangle$

Then this integrand is also seen as

$$\int_C M\,dx + N\,dy + P\,dz \quad := \int_a^b \left\{ M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + P(\vec{r}(t))\,z'(t)\right\} dt$$

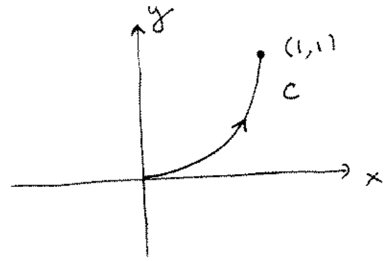
$dx = x'(t)\,dt$

$dy = y'(t)\,dt$

$dz = z'(t)\,dt$

Example

Let C be the parabola $\underset{0 \le x \le 1}{y = x^2}$, going from $(0,0)$ to $(1,1)$



Let $\vec{F}(x,y) = \langle 2x+y, -x \rangle$

Compute $\int_C \vec{F} \cdot d\vec{r} = \int_C (2x+y)\,dx - x\,dy$

Method 1 $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$

Method 2 $\vec{r}(x) = \langle x, x^2 \rangle$ $0 \le x \le 1$

(really method 1 all over, but explains dual notation for line ints)

Method 3 $\vec{r}(y) = \langle \sqrt{y}, y \rangle$