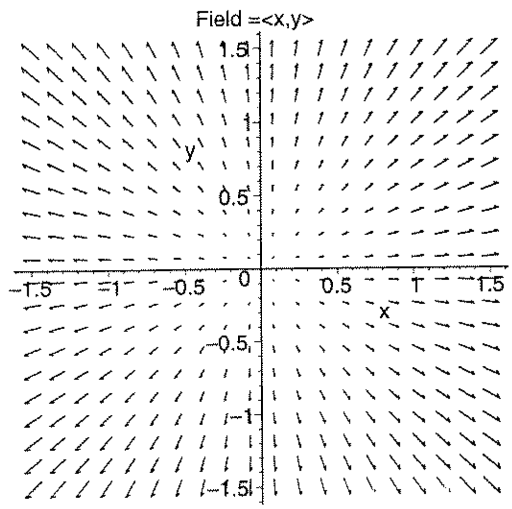
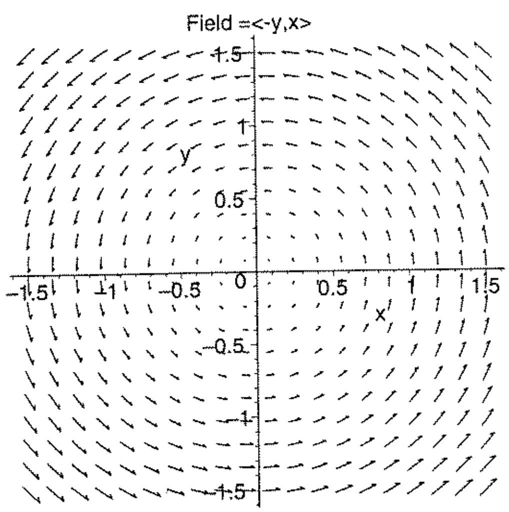
Math 2210-4
Monday 4/11

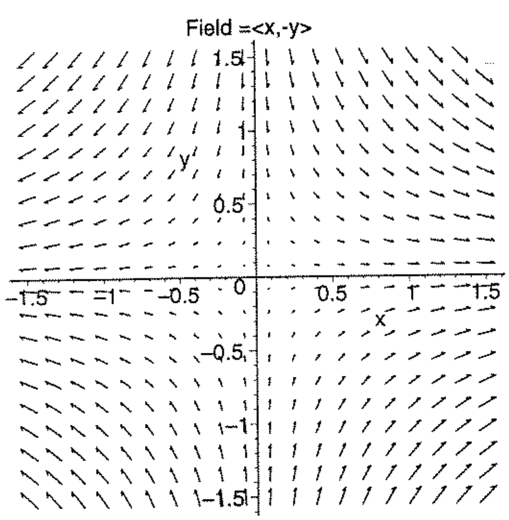
§17.1 Vector fields:
these arise naturally in physics as force fields, also as velocity fields in fluid motion



$\vec{F} = \langle M(x,y), N(x,y) \rangle = \langle x, y \rangle$
gradient field? $(\vec{F} = \nabla f)$



$\langle M, N \rangle = \langle -y, x \rangle$
gradient field?
(Think Escher!)



$\langle M, N \rangle = \langle x, -y \rangle$
gradient field?

Think of ∇ meaning $\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle = \langle \partial_x, \partial_y, \partial_z \rangle$ in $\mathbb{R}^3$

$\boxed{\text{divergence}}$ or $\langle \partial_x, \partial_y \rangle$ in $\mathbb{R}^2$

$\downarrow$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle M, N, P \rangle = M_x + N_y + P_z \quad \text{in } \mathbb{R}^3$$

$$= \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y} \rangle \cdot \langle M, N \rangle = M_x + N_y \quad \text{in } \mathbb{R}^3$$

$\boxed{\text{curl}}$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \quad \text{in } \mathbb{R}^3$$

in $\mathbb{R}^2$, scalar curl is $N_x - M_y$
curl $\vec{F}$

<u>note</u> if $\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle$, then $\operatorname{curl} \vec{F} = \langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle$

$$= \vec{0}\,!$$

i.e. $\nabla \times (\nabla f) = 0\,!$

Compute div, curl for our 3 $\mathbb{R}^2$ vector fields. Think about what they may mean.
(in relation to the English meaning of <u>divergence</u> and <u>curl</u>)

So, curl $\vec{F}$ measures whether $\vec{F}$ could be a gradient vector field.
It <u>also</u> measures "rotation" in a vector field,
i.e. suppose the vector field is a field of tangent vectors for moving particles, as in a fluid flow.
It turns out curl $\vec{F}$ is a vector which measures how much they're rotating.

e.g. $\vec{F} = \langle -y, x, 0 \rangle$

$\vec{\nabla} \times \vec{F} =$

div $\vec{F}$ measures whether the vector field illustrates expansion or contraction

Chapter 17 is largely about making these heuristic descriptions precise,
and is part of the mathematical foundation for many areas of physics

Example

$\vec{F} = \langle 2x+y, e^x, 2z \rangle$

Compute

$\vec{\nabla} \times \vec{F}$

$\vec{\nabla} \cdot \vec{F}$

Example in $\mathbb{R}^3$, $f(x,y,z) = \dfrac{-1}{\sqrt{x^2+y^2+z^2}}$ $\left(= -\frac{1}{e} \right)$.

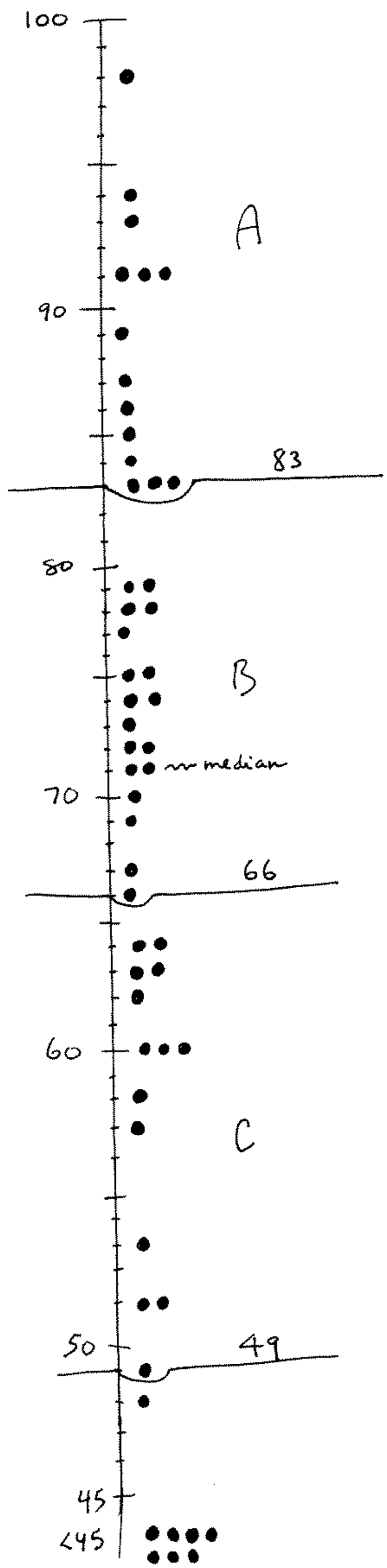
Compute $-\nabla f = \vec{F}$

notice this is the inverse square law (of gravity & ~~electrostatics~~ electric charge).

Thus $\vec{\nabla} \times \vec{F} = \vec{0}$.

Show also $\vec{\nabla} \cdot \vec{F} = 0$ (in HW).

Math 2210-4
Exam 2 Scores
$n = 54$



4) In this problem we will consider the function $f(x,y) = e^{(2x)}(2+\sin(y))$.

4a) Compute the gradient of f(x,y). Then verify that its value at (0,0) is the vector < 4, 1 >. (5 points)

$\nabla f = \langle f_x, f_y \rangle = \cancel{\langle 4e^{2x}, e^{2x}\cos y\rangle}\ \langle 2e^{2x}(2+\sin y), e^{2x}\cos y \rangle$

$\nabla f(0,0) = \langle 4, 1 \rangle$

4b) In what unit direction is f(x,y) increasing most rapidly at the origin? What is the rate of change of f in this direction? (5 points)

in direction of ∇f : $\vec{u} = \frac{1}{\sqrt{17}}\langle 4,1\rangle$

rate of change $D_{\vec{u}} f(0,0) = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = |\nabla f| = \sqrt{17}$

4c) Use the differential (or tangent) approximation to approximate f(0.1, -0.2), using f(0,0)=2. (5 points)

$dz = f_x\,dx + f_y\,dy$

$dx = .1$
$dy = -.2$
$f_x = 4$
$f_y = 1$

$dz = 4\cdot(.1) + 1(-.2) = .2$

so $f(.1,-.2) \approx f(0,0) + dz = \boxed{2.2}$

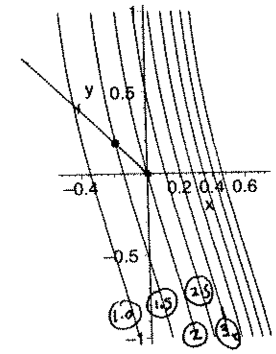
4d) Compute the directional derivative $D_u f(0,0)$, where **u** is the unit vector in the same direction as the vector <-1, 1 >. (5 points)

$D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$= \langle 4,1\rangle \cdot \frac{1}{\sqrt{2}}\langle -1,1\rangle$

$= \frac{-4+1}{\sqrt{2}} = \boxed{\frac{-3}{\sqrt{2}}} \approx -2.12$

Here is a picture of some level curves of f(x,y). They are labeled by the value of f along each curve.



4e) Use the level curve picture and an index card ruler to estimate the same directional derivative you just computed exactly in (4d). Make sure to show/explain your work! (Your estimate may only be with 10 or 20% of the exact value, because you're not working on a very magnified scale.) (5 points)

Using the two highlited points along the $\langle -1,1\rangle$ direction

estimate $\frac{\Delta f}{\Delta s} = \frac{-.5}{\approx .27} \approx -1.85$

(compared to exact value of $-\frac{3}{\sqrt{2}} \approx -2.12$)

4f) Let **r**(t) be the vector-valued function $\mathbf{r}(t) = \langle -e^{(2t)}, -\sin(2t) \rangle$. Use the multivariable chain rule to compute the t-derivative of the composition f(**r**(t)), at t = 0. (10 points)

$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

$\vec{r}(0) = \langle -1, 0 \rangle$

$\vec{r}\,'(t) = \langle -2e^{2t}, -2\cos 2t \rangle$

$\vec{r}\,'(0) = \langle -2, -2 \rangle$

$\cancel{\nabla f(0,0) = \langle 4,1\rangle}$ $\nabla f(-1,0) = \langle 4e^{-2}, e^{-2} \rangle$ (see 4a)

$\langle 4e^{-2}, e^{-2} \rangle \cdot \langle -2,-2 \rangle = -10e^{-2}$

Corrected sol'n!

5