Math 2210-1
Wed 8 Sept

HW for Wed 9/15 (more to be assigned on Fri!) ①
§11.5 1, 11, 15, 17, 22, 28, 29, 33

§11.5 Linear systems and matrices

General set up for m linear equations in n-unknowns:
(goal is to find all n-tuples $(x_1, x_2, \ldots x_n)$ which solve each eqtn (simultaneously))

$$\boxed{\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \cdots & & + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots & & + a_{2n}x_n = b_2 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots & & + a_{mn}x_n = b_m. \end{array}}$$

If you have 3 unknowns (x_1, x_2, x_3), or (x, y, z) you are looking for intersections of planes in $\mathbb{R}^3$
If you have 2 unknowns, e.g. (x,y), you are looking for intersections of lines in $\mathbb{R}^2$

BUT linear systems occur in very diverse settings, even though you may give them this geometric interpretation

example 1

actual eqtns

$$\begin{aligned} 3x + y &= 1 \\ x - 2y &= 5 \end{aligned}$$

$$\begin{array}{ll} E_2 & x - 2y = 5 \\ E_1 & 3x + y = 1 \end{array}$$

$$\begin{array}{ll} E_1 & x - 2y = 5 \\ -3E_1 + E_2 & 7y = -14 \end{array}$$

$$\begin{array}{ll} & x - 2y = 5 \\ E_2/7 & y = -2 \end{array}$$

$$\begin{array}{ll} E_1 + 2E_2 & x \quad = 1 \\ & y = -2 \end{array}$$

synthetic version

$$\left[\begin{array}{cc|c} 3 & 1 & 1 \\ 1 & -2 & 5 \end{array}\right]$$

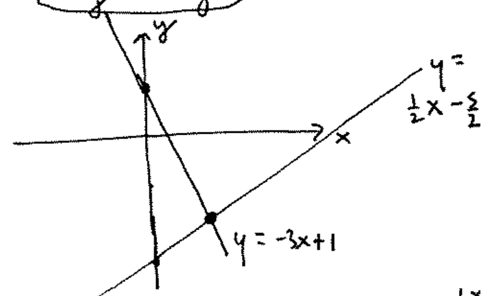
$$\begin{array}{l} R_2 \\ R_1 \end{array} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 3 & 1 & 1 \end{array}\right]$$

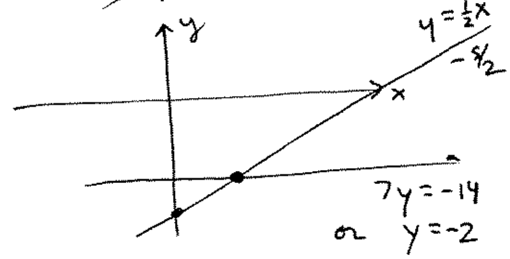
$$\begin{array}{l} R_1 \\ -3R_1 + R_2 \end{array} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 7 & -14 \end{array}\right]$$

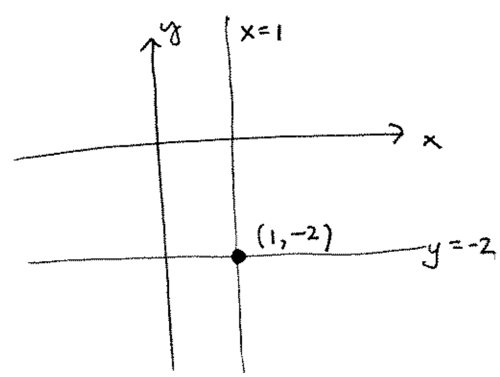
$$\begin{array}{l} \\ R_2/7 \end{array} \left[\begin{array}{cc|c} 1 & -2 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

$$\begin{array}{l} R_1 + 2R_2 \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$$

geometry







$\boxed{\text{Elementary equation operations:}}$ $\xrightarrow{\text{synthetic}}$ $\boxed{\text{Elementary row operations}}$

1. interchange ("swap") two eqtns
2. multiply an eqtn by a non-zero constant
3. replace an equation with the sum of itself and a multiple of another eqtn

1. swap rows
2. multiply a row by a non-zero const.
3. replace a row with the sum of itself and another row.

Theorem : Consider the set of solutions to a system of equations

$$\begin{array}{l} E_1 \\ E_2 \\ \vdots \\ E_m \end{array}$$

The solution set remains the same whenever we use elementary equation operations to construct a new system of equations (even though the ~~s~~ solution set to an individual eqtn may change.).

proof: (by the way, this theorem is illustrated by the example on page 1).

↳ After you plug in the variables you are just dealing with the question of whether real numbers are equal. The theorem then follows because for real numbers the following pairs of eqtns are all equivalent:

$$\begin{array}{llll} c_1 = c_2 & d_1 = d_2 & tc_1 = tc_2 \quad t \neq 0 & c_1 = c_2 \\ d_1 = d_2 & c_1 = c_2 & d_1 = d_2 & d_1 + tc_1 = d_2 + tc_2 \end{array}$$

Example 2

$$\begin{aligned} x - 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ -x + z &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ -1 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1+R_2 \\ R_1+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 14 & 4 & 8 \\ 0 & -2 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ R_2/2 \\ R_3/-2 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 7 & 2 & 4 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\begin{array}{r} \\ R_3 \\ R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 7 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -7R_2+R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 9 & 18 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

← row echelon form

can backsolve from here

$z = 2$

$\cancel{y} - z = -2 \Rightarrow y = \cancel{}\ z - 2 = 0$

$x = 4 + 2y - z = 4 - 2 = 2$

$x = 2$
$y = 0$
$z = 2$

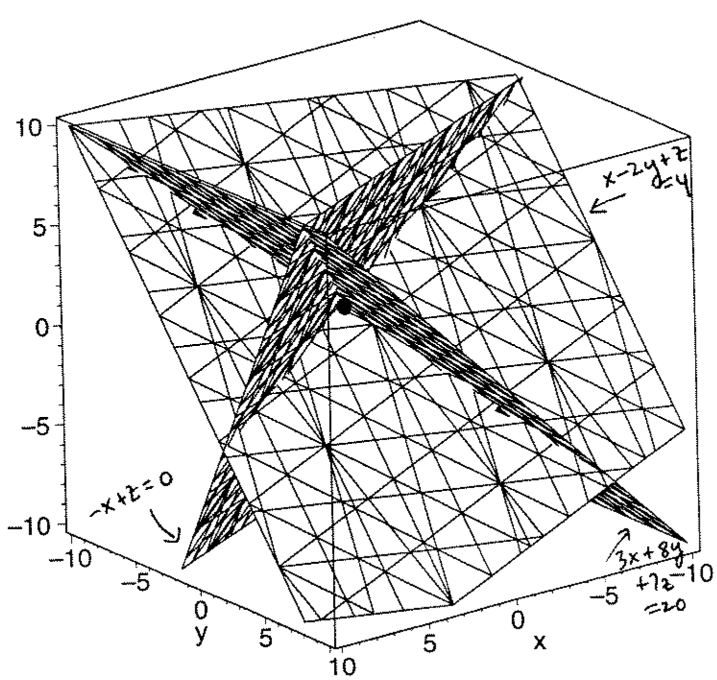
$$\begin{array}{r} R_1-R_3 \\ R_3+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

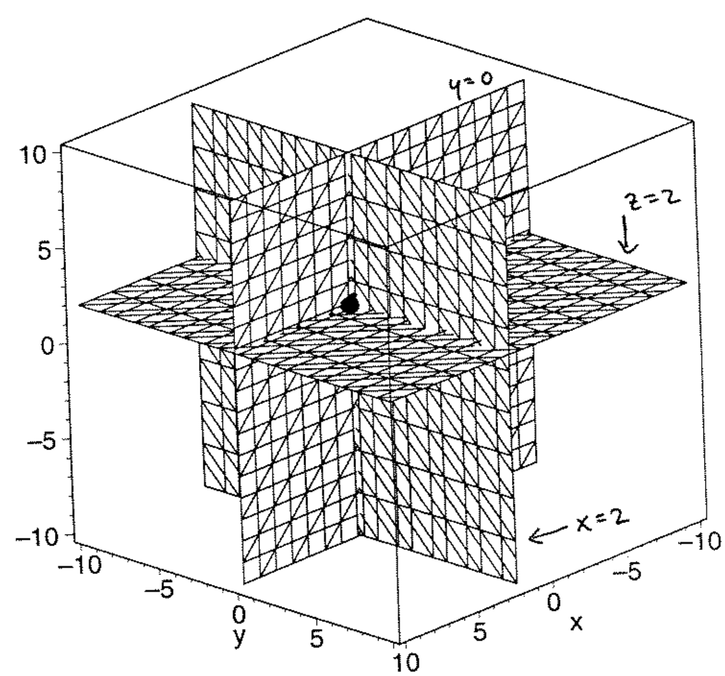
$$\begin{array}{r} 2R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

← reduced row echelon form

see $x = 2$, $y = 0$, $z = 2$ immediately

check $x=2$, $y=0$, $z=2$ solves original system!





~~@~~ Draw schematic pictures!

lots of possibilities!

example 3

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 0 & 14 & 4 & 8 \end{array}\right]$$

$$\begin{array}{l} \\ -3R_1+R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 14 & 4 & 8 \\ 0 & 14 & 4 & 8 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3-R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 7 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & 2/7 & 4/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

~~zz t tang~~

$$\begin{array}{l} 2R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 11/7 & 36/7 \\ 0 & 1 & 2/7 & 4/7 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$y = 4/7 - 2/7\, t$

$x = 36/7 - 11/7\, t$

$\langle x, y, z\rangle = \langle 36/7, 4/7, 0\rangle + t\langle -11/7, -2/7, 1\rangle$

what geometric object is this?

example 4

or

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 0 & 14 & 4 & 7 \end{array}\right] \leftarrow \text{only thing changed.}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 14 & 4 & 8 \\ 0 & 14 & 4 & 7 \end{array}\right] \quad \text{(as before)}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 14 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{array}\right] \leftarrow 0 = -1\,!\ \text{NO SOLUTION.}$$

example 5

or $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 4 \end{array}\right]$

$z = t$

$y = s$

$x = 4 - z + 2y = 4 - t + 2s$

$\langle x, y, z\rangle = \langle 4, 0, 0\rangle + s\langle 2, 1, 0\rangle + t\langle -1, 0, 1\rangle$

parametric expression of a plane!