

Quadric Surface Zoo
Math 2210-1
Monday, September 27, 2004

Quadric surfaces are collections of points in R^3 which satisfy a quadratic equation in the x-y-z variables. Thus, these surfaces are defined implicitly, and satisfy the next simplest implicit equations after the linear equations which define planes. It is good to draw (quadric) surfaces for yourself - the process of learning to draw them is related to your ability to visualize them. This document is Maple output, and is posted on our lecture page. If you open it in Maple, you can modify the surfaces, and also manipulate them with your mouse. You'll try to draw these surfaces in class, by hand.

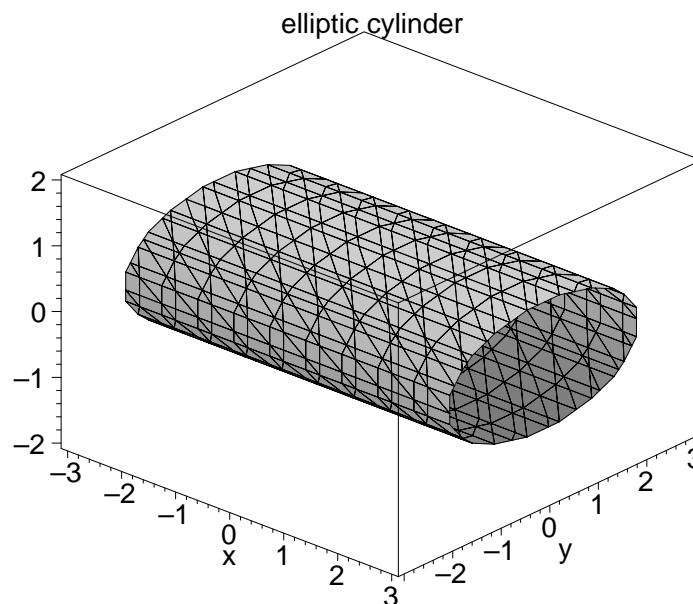
It turns out from linear algebra that even if your quadratic equation has cross terms (xy,xz,yz), there is a rotated coordinate system in which the solution set satisfies a quadric surface equation without cross terms. In our section of 2210, we'll only worry about "no-cross term" surfaces. If you eventually take Math 2270 (Linear Algebra) you should study this question in detail. Our text discusses these issues briefly in section 12.3 and 11.7.

```
[ > with(plots): #the library of graphing commands
```

```
(0) We already understand the solution set to equations ax+by+cz=d, i.e. planes
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(1) Cylinders: If one of the variables is missing from your equation, then that variable is free to be anything, as long as the other two variables satisfy the equation. Thus, if a point lies on your surface, the entire line of points you get by varying the missing variable is also on your surface - we call such surfaces cylinders even if the cross section is not circular. If you can recognize the curve in the plane of the two visible variables, you can draw the cylindrical surface in 3-space.
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```
> implicitplot3d(z^2+y^2/4=1,x=-3..3,y=-3..3,z=-2..2,
  axes=boxed,
  title='elliptic cylinder');
```



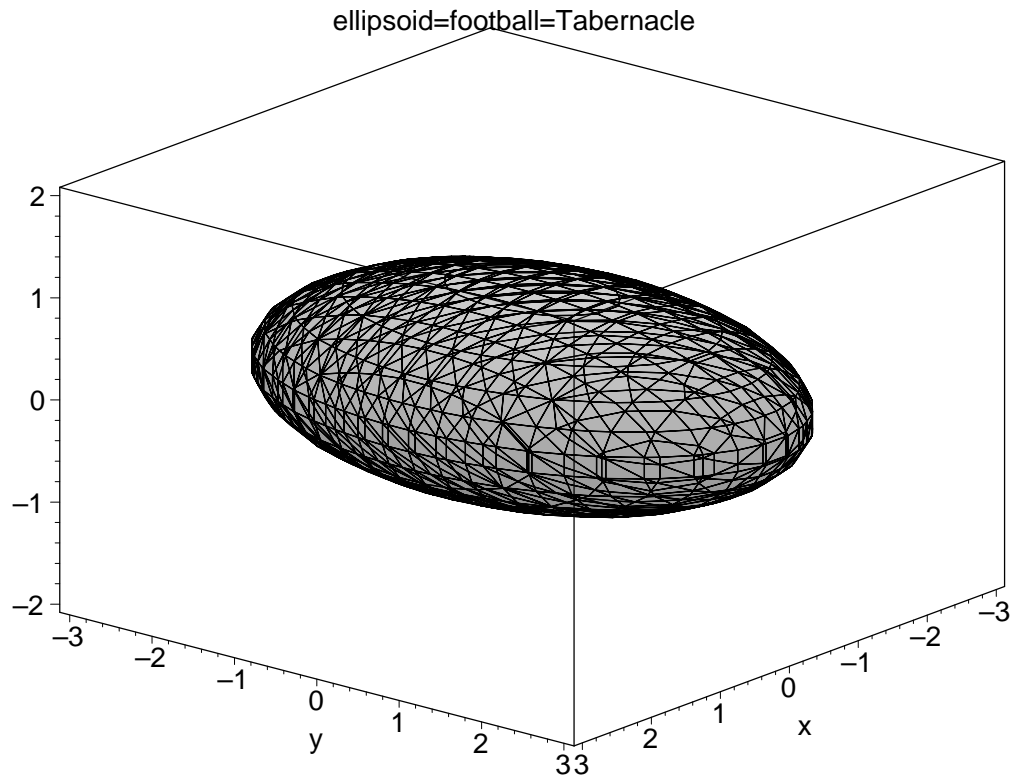
Thus, all of your knowledge of conics (ellipses, parabolas, hyperbolas) translates into knowledge of cylindrical quadric surfaces.

2) Ellipsoids: An ellipsoid centered at the origin will be the solution set to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

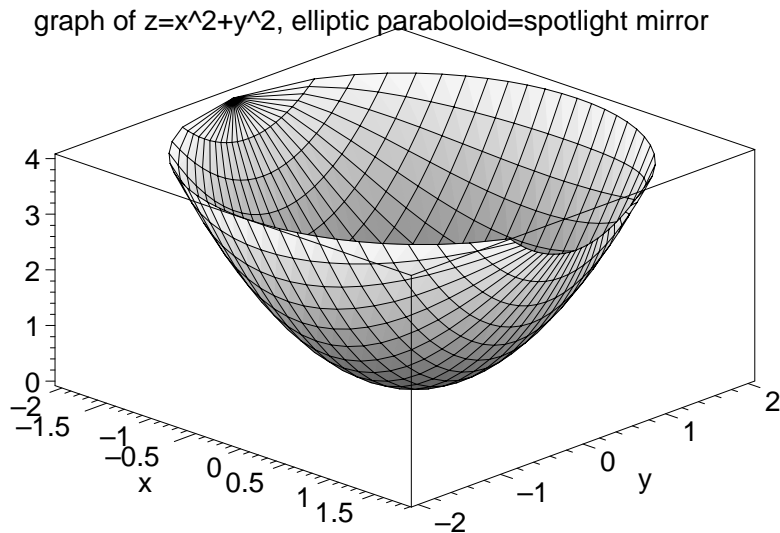
If you look in each coordinate plane, by requiring one of your variables to be zero, you get a trace curve which is an ellipse, and this is the key to drawing the ellipsoid by hand.

```
> implicitplot3d(x^2/4+y^2/9+z^2=1,x=-3..3,y=-3..3,z=-2..2,  
  grid=[20,20,20],axes=boxed,title='ellipsoid=football=Tabernacle');
```

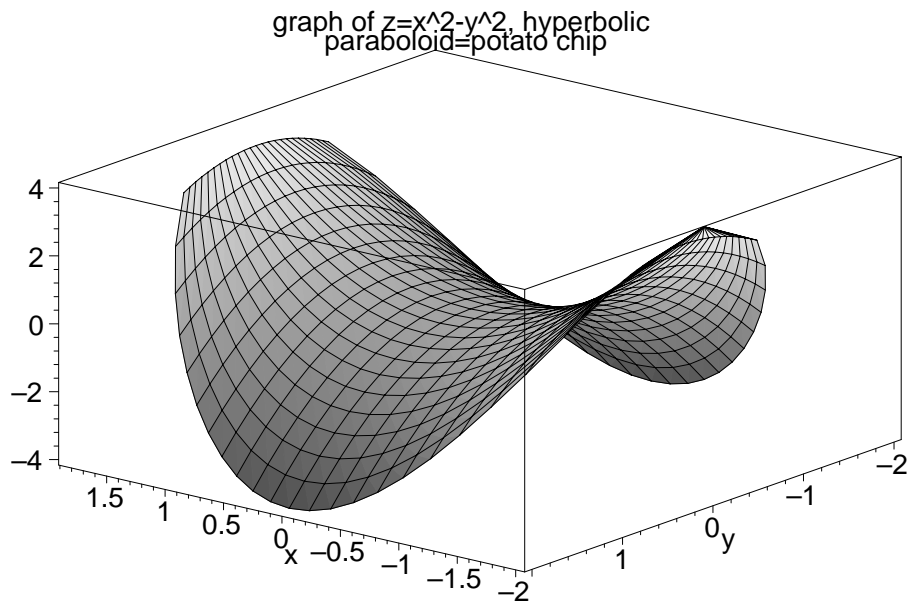


3) Paraboloids: If one of your variables (say "z") only appears linearly, but the other two appear quadratically, then your creature is a paraboloid. Depending on whether the cross sections perpendicular to the linear variable (z-) axis are ellipses or hyperbolas, you either have an elliptic or hyperbolic paraboloid.

```
> plot3d(x^2+y^2,x=-2..2,y=-sqrt(4-x^2)..sqrt(4-x^2),axes=boxed,
  title='graph of z=x^2+y^2, elliptic paraboloid=spotlight mirror');
```



```
> plot3d(x^2-y^2,x=-2..2,y=-sqrt(4-x^2)..sqrt(4-x^2),axes=boxed,
  title='graph of z=x^2-y^2, hyperbolic paraboloid=potato chip');
```



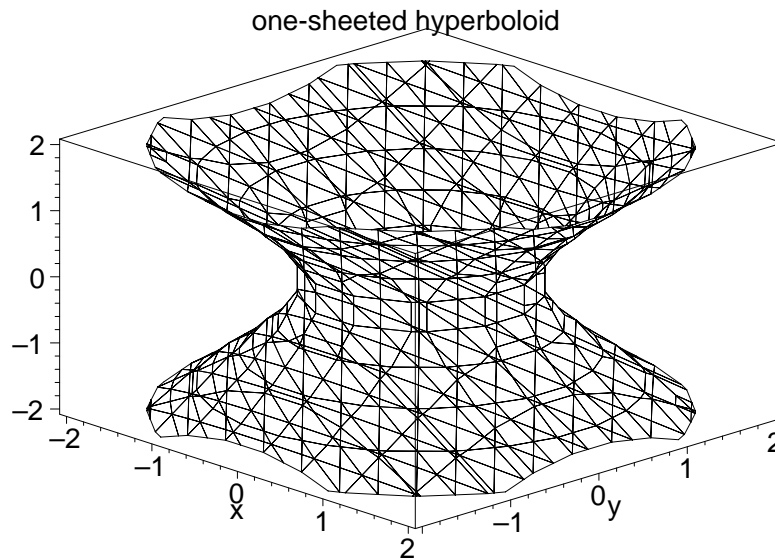
4) Hyperboloids: If all variables appear quadratically, but not all with the same sign - that was the case of ellipsoids, then there are several interesting possibilities.

4a) One-sheeted hyperboloid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Notice that the traces of this surface in horizontal planes are all non-empty, and are ellipses. Thus there is only one "piece" to this surface.

```
> implicitplot3d(x^2+y^2-z^2=1,x=-2..2,y=-2..2,z=-2..2,
  axes=boxed,
  title='one-sheeted hyperboloid');
```



4b) Two-sheeted hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

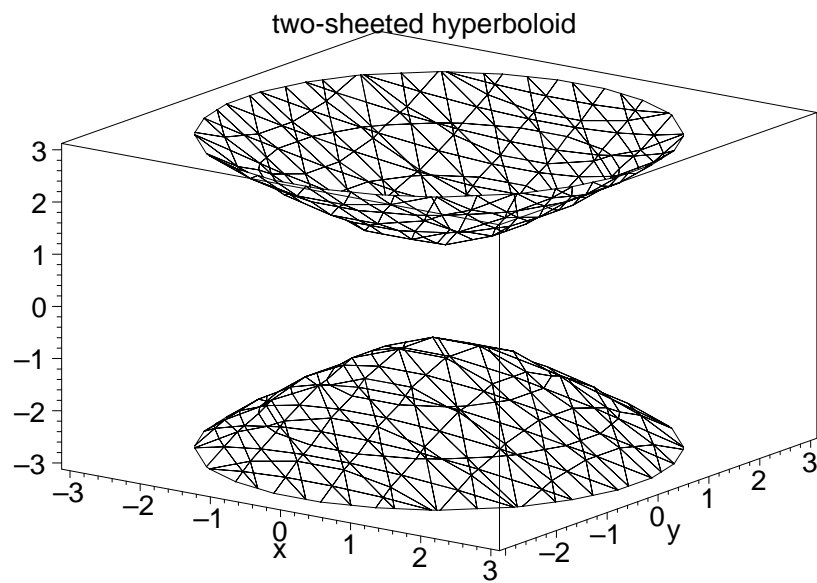
Now, when you look at traces in horizontal planes $z=\text{constant}$, the value of the constant must make

$$-1 + \frac{z^2}{c^2}$$

positive in order for the cross section to be an ellipse. If the value is negative then there is NO trace.

Thus the surface is a union of two pieces, on one of which z is at least c and on the other z is less than $-c$.

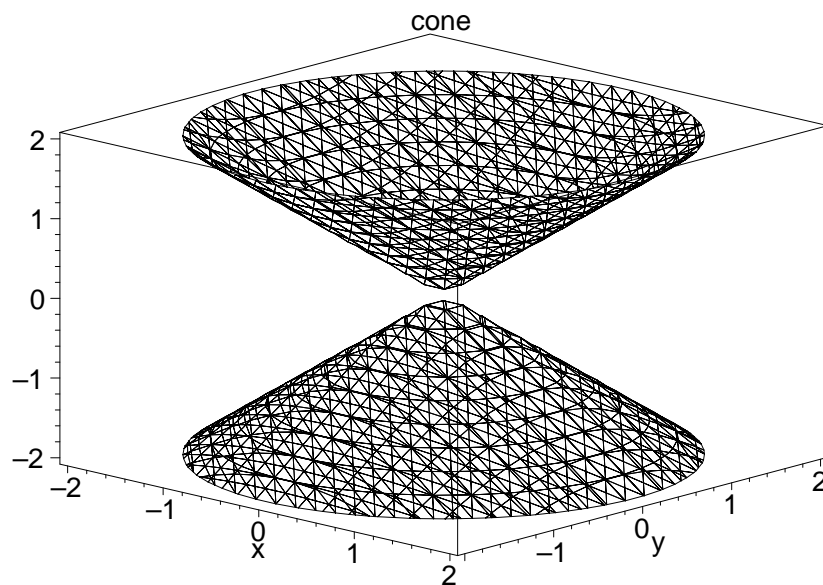
```
> implicitplot3d(x^2+y^2-z^2=-1,x=-3..3,y=-3..3,z=-3..3,
  axes=boxed,title='two-sheeted hyperboloid');
```



4c) Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

```
> implicitplot3d(x^2+y^2-z^2=0,x=-2..2,y=-2..2,z=-2..2,
  grid=[20,20,20],axes=boxed,title='cone');
```



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