

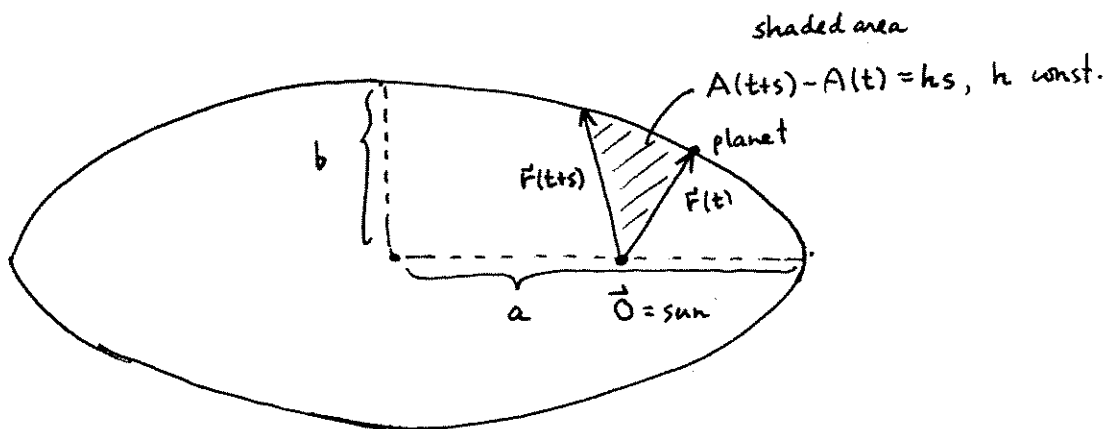
Math 2210-1

Fri 9/24

Kepler's Laws and Newton's mathematical model: the inverse square law of gravitational attraction
[these notes follow ideas in notes of Hugo Rossi ~ don't exactly follow an text]

descriptive, based on painstaking observations of planetary motion: from $\vec{O}$ = sun location

- KEPLER
- ① The orbit of a planet (a curve with position vector $\vec{r}(t)$, t = time) is an ellipse lying in a fixed plane. The sun is at a focus of the ellipse.
 - ② Equal areas are swept out in equal times; if $A(t)$ is area swept out by $\vec{r}(t)$ (after some initial time t_0), then $A(t+s) - A(t) = hs$, h constant. i.e. $A'(t) \equiv h$, constant. (Kepler didn't know calculus, though, that was another discovery of Newton.)
 - ③ Let T = period of orbit. Let a = $\frac{1}{2}$ length of major axis for ellipse. Then $T = \tilde{c} a^{3/2}$ ($T^2 \sim a^3$), with the constant $\tilde{c}$ independent of the planet (but it may depend on the "sun")



Newton derived a model to explain Kepler, and this is also the origin of his discovery of the inverse square law of gravitational attraction:

$$|\vec{F}| = \frac{GmM}{r^2}$$

G = univ constant

m, M = object masses

$r = |\vec{r}|$ = separation distance

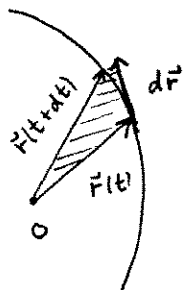
Newton:

- NEWTON
- ① An orbiting planet orbits in a fixed plane containing the sun, and satisfies the equal area condition ② if and only if the acceleration is centripetal ($\parallel \vec{r}$)
 - ② The only centripetal acceleration consistent with ① is one with magnitude $\frac{c}{r^2}$ (inverse square law), c = constant, for a given planet
 - ③ ③ holds if and only if the constant $\tilde{c}$ in ② is the same for each planet's orbit (= GM , as it turns out.)

This discussion incorporates almost all of the ideas we've talked about so far in 2210! (In addition to being a historical breakthrough in physics & math.)

(2)

Lemma (1a) If the orbit lies in a plane containing the sun and if equal area (II) holds, then the acceleration is centripetal:



in time dt ,

$$dA = \frac{1}{2} |\vec{r} \times d\vec{r}|$$

$$= \frac{1}{2} |\vec{r} \times \vec{r}'(t) dt|$$

(triangle area)
geometric meaning of cross product!
and
affine (tangent line)
(differential)
approximation!

(1.1)
$$\frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \vec{r}'|$$

If $\vec{r}$, hence $\vec{r}'$ lie in plane thru sun, then $\vec{r} \times \vec{r}'$ is a multiple of this plane's unit normal vector " $\vec{k}$ ".

If $\frac{dA}{dt} \equiv h$ const (II),

(since $\vec{r} \times \vec{r}' \perp \vec{r}$ and $\vec{r}'$)

the multiple of $\vec{k}$ which $\vec{r} \times \vec{r}'$ is, is constant, i.e.

(1.2)
$$\vec{r} \times \vec{r}' = 2h \vec{k}$$
 (if we orient $\vec{k}$ as usual)

Thus

(1.3)
$$\frac{d}{dt} (\vec{r} \times \vec{r}') \equiv \vec{0}$$

$$\parallel$$

$$\vec{r}' \times \vec{r}' + \vec{r} \times \vec{r}''$$

$$= 0$$

(1.4)
$$\vec{r} \times \vec{r}'' \equiv \vec{0}$$
 i.e. $\vec{r}'' \parallel \vec{r}$, centripetal accel. $\blacksquare$

Lemma (1b) If the acceleration is centripetal, then the planet orbits the sun in a fixed plane containing the sun, and the equal area condition (II) holds

proof: if $\vec{r}'' \parallel \vec{r}$ then (1.4) holds, so also

(1.3):
$$\frac{d}{dt} (\vec{r} \times \vec{r}') \equiv \vec{0}$$

antidifferentiating yields $\vec{r} \times \vec{r}' = \vec{K}$, $\vec{K}$ constant
in particular $\vec{r} \cdot \vec{K} \equiv 0$

so $\vec{r}(t)$ lies in the plane $\langle x, y, z \rangle \cdot \vec{K} = 0$,
which contains the origin (= sun).

choose coords so that the plane is the x-y plane,
and $\vec{K} = 2h \hat{k}$ $h > 0$ const

Thus (1.2) holds, and (1.1) implies $\frac{dA}{dt} \equiv h$ $\blacksquare$

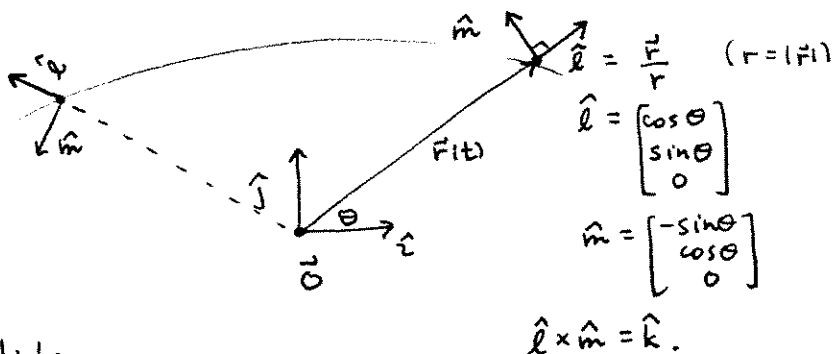
To prove Newton ②③ we introduce a new "frame" (moving basis)

old, fixed frame:

plane is x-y plane, basis $\hat{i}, \hat{j}$
normal vector $\hat{k}$

planet orbits counterclockwise so $\vec{r} \times \vec{r}'$ is positive multiple of $\hat{k}$.

moving frame
along $\vec{r}(t)$ given
by $(\hat{\ell}, \hat{m}, \hat{k})$



Using r and θ to describe particle location we notice

Lemma 2

$$(i) \quad \frac{dA}{dt} = \frac{r^2}{2} \frac{d\theta}{dt} \quad (dA = \frac{r^2}{2} d\theta)$$

"
 h

, so $\boxed{\theta' = \frac{2h}{r^2}}$

$$(ii) \quad \vec{r} = r \hat{\ell}, \text{ so } \vec{r}' = r' \hat{\ell} + r \hat{\ell}'$$

$$(iii) \quad \hat{\ell}' = \theta' \hat{m}$$

$$(iv) \quad \hat{m}' = -\theta' \hat{\ell}$$

Proof of Newton ②③:

④

centripetal acceleration

$$\Rightarrow \vec{r}'' = -f \hat{\ell} \quad , \text{ where a priori } f \text{ could depend on anything, } \\ \text{e.g. time, } \vec{r}, \dots$$

$$\begin{aligned} \Rightarrow \vec{r}'' \times \hat{k} &= -f \hat{\ell} \times \hat{k} \\ &= f \hat{m} \\ &= f \frac{1}{\theta'} \hat{\ell}' \quad \text{Lemma 2.iii} \end{aligned}$$

$$(3.1a) \quad \boxed{\vec{r}'' \times \hat{k} = f \frac{r^2}{2h} \hat{\ell}'} \quad \text{Lemma 2i}$$

If $f = f(r) = \frac{c}{r^2}$ [this is for half of Newton's thm]

$$(3.1b) \quad \Rightarrow \boxed{\vec{r}'' \times \hat{k} = \frac{c}{2h} \hat{\ell}'} \quad [\text{the } r^2\text{'s cancel}]$$

$$\Rightarrow \vec{r}' \times \hat{k} = \frac{c}{2h} \hat{\ell} + \vec{E} \quad \vec{E} \text{ is constant of integration, is } \perp \hat{k} \text{ since all the other terms are.}$$

$$(3.2) \quad \boxed{\begin{matrix} (r' \hat{\ell} + r \hat{\ell}') \times \hat{k} = \frac{c}{2h} \hat{\ell} + \vec{E} \\ \quad \quad \quad \downarrow \\ \quad \quad \quad \theta' \hat{m} \end{matrix}}$$

dot both sides with $\hat{\ell}$:

$$r \theta' = \frac{c}{2h} + \vec{E} \cdot \hat{\ell}$$

Lemma 2.i

$$\frac{2h}{r} = \frac{c}{2h} + \vec{E} \cdot \hat{\ell}$$

$$\frac{r}{2h} = \frac{1}{\frac{c}{2h} + \vec{E} \cdot \hat{\ell}}$$

(3.3)

$$\boxed{r = \frac{4h^2/c}{1 + \frac{2h}{c} \vec{E} \cdot \langle \cos \theta, \sin \theta \rangle}}$$

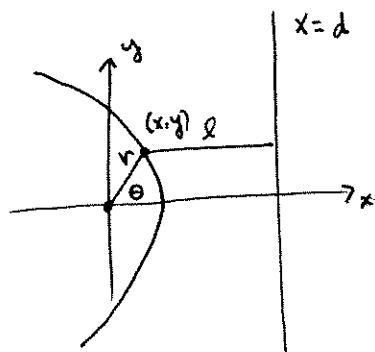
this is a polar coord eqn for something!

Notice, we can choose our original $\hat{i}, \hat{j}$ so that $\vec{E} = \lambda \hat{i}$, so

(3.4)

$$\boxed{r = \frac{4h^2/c}{1 + \frac{2h\lambda}{c} \cos \theta}}$$

Conic sheet:



$\frac{r}{d-x} = e$ constant, called eccentricity

$$\frac{r}{d-r\cos\theta} = e$$

$$r = e(d-r\cos\theta)$$

$$r(1+e\cos\theta) = ed$$

$$r = \frac{ed}{1+e\cos\theta}$$

$e < 1$ ellipse
($e=0$ circle)
 $e=1$ parabola
 $e > 1$ hyperbola

So the 2nd page 4 box says we have a conic, with

$$e = \frac{2h\lambda}{c}, \quad d = \frac{2h}{\lambda}$$

To explain Kepler (III) ($T^2 \propto a^3$), derive cartesian eqn of ellipse:

$$\frac{r}{d-x} = e$$

$$\sqrt{x^2+y^2} = e(d-x)$$

$$x^2+y^2 = e^2(x^2-2dx+d^2)$$

$$x^2(1-e^2) + 2de^2x + y^2 = e^2d^2$$

$$(1-e^2)\left[x^2 + \frac{2de^2}{1-e^2}x\right] + y^2 = e^2d^2$$

$$(1-e^2)\left[x + \frac{de^2}{1-e^2}\right]^2 + y^2 = e^2d^2 + \frac{d^2e^4}{1-e^2}$$

$$= \frac{e^2d^2}{1-e^2}$$

$$\frac{\left(x + \frac{de^2}{1-e^2}\right)^2}{\left(\frac{ed}{1-e^2}\right)^2} + \frac{y^2}{\left(\frac{ed}{\sqrt{1-e^2}}\right)^2} = 1$$

$\uparrow$ a^2 $\uparrow$ b^2

$$\text{so } b = \sqrt{a} \sqrt{ed} = \sqrt{a} \frac{2h}{\sqrt{c}}$$

Why is this ellipse area?
↓
(Think "Bob")

now, $\frac{dA}{dt} = h$, so $\int_0^T h dt = \text{ellipse area} = \pi ab$

$$hT = \pi ab$$

$$T = \frac{\pi ab}{h} = \frac{\pi a \sqrt{a} \frac{2h}{\sqrt{c}}}{h} = \frac{\pi}{\sqrt{c}} a^{3/2}$$

Kepler (III)

c came from $f(r) = \frac{c}{r^2}$

By Kepler (III), c doesn't depend on planet.

We've shown $f(r) = \frac{c}{r^2}$ ^{necessarily} (same c for each planet)

implies all of Kepler's laws

Can any other centripetal force imply these laws?

Start with

$$r = \frac{ed}{1 + e \cos \theta} \quad \frac{dA}{dt} = h$$

define $\lambda = \frac{2h}{d}$, $c = \frac{2h^2}{e}$

gives box on page 5,

so lower box on page 4, (3.4) holds

work backwards on page 4.

You can verify (3.2) by checking the $\hat{m}$ component, since $\hat{\ell}$ components check automatically.

get to (3.1b)

$$\vec{r}'' \times \vec{k} = \frac{c}{2h} \hat{\ell}'$$

but since acceleration is centripetal, we also know

$$(3.1a) \quad \left| \vec{r}'' \times \vec{k} = f \frac{r^2}{2h} \hat{\ell}' \right| \rightarrow \text{so } f \text{ must equal } \frac{c}{r^2}$$