Math 2210
Wed 22 Sept

Homework for Monday 9/27

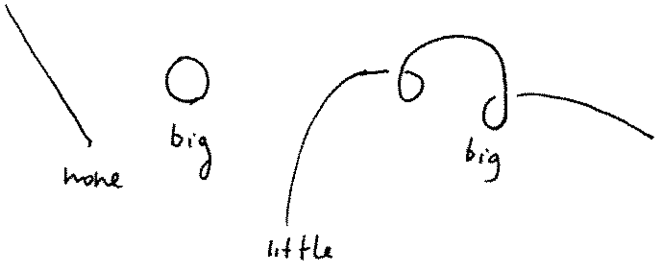
12.2 (7, 8, 16, 17, 25) — sketch this spiral show $\vec{r}(\pi/2)$, $\vec{T}, \vec{N}, \vec{a}$ illustrating tang & normal part of $\vec{a}$

(33, 43, 50, 55)

Exam Wed 9/29 thru 12.2

Fri: Kepler/Newton
Mon 12.3

§12.2 curvature & acceleration



curvature is a quantitative measure of how "fast" a curve is bending.

w.r.t. ?
how about, if you traverse the curve with unit speed ($|\vec{r}'(t)| \equiv 1$).

digress:
unit speed reparameterizations:

example: $\vec{r}(t) = \begin{bmatrix} 10\cos t \\ 10 \sin t \end{bmatrix}$, $0 \le t \le 2\pi$
parameterizes the circle $x^2+y^2=100$ of radius 10 centered at the origin

$\vec{r}'(t) = \begin{bmatrix} -10 \sin t \\ 10 \cos t \end{bmatrix}$

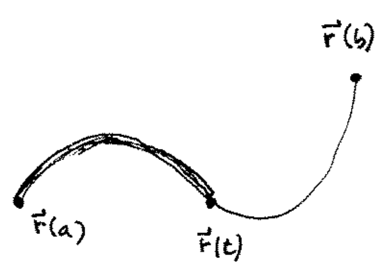
$|\vec{r}'(t)| = 10$, so we're going 10 times too fast

if we let $s = 10t$; $t = \frac{s}{10}$

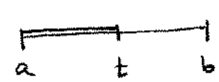
then $\vec{\rho}(s) = \vec{r}(t(s)) = \begin{bmatrix} 10 \cos\frac{s}{10} \\ 10 \sin \frac{s}{10} \end{bmatrix}$ satisfies $|\vec{\rho}'(s)| = 1$!

In general, if
$\vec{r}: [a,b] \to \mathbb{R}^n$

Define $s(t) = \int_a^t |\vec{r}'(\tau)|\, d\tau$ = length of arc of curve between $\vec{r}(a)$ and $\vec{r}(t)$



1210 FTC $\Rightarrow$ $\boxed{\frac{ds}{dt} = |\vec{r}'(t)| = v(t)}$



(makes sense: how fast the arc length is changing equals speed)

Assuming $v(t) > 0$ on $[a,b]$, $s(t)$ is a strictly increasing function, $s(a)=0$, $s(b)=L$ total length
so by inverse fcn thm [e.g. recall ln, exp in Calculus]
there is an inverse function $t(s)$, $t:[0,L] \to [a,b]$,
and $\vec{\rho}(s) := \vec{r}(t(s))$ is unit speed, since $\vec{\rho}'(s) = \frac{d\vec{r}}{dt}\frac{dt}{ds}$ (chain rule)

$|\vec{\rho}'(s)| = \left|\frac{dt}{ds}\right| \left|\frac{d\vec{r}}{dt}\right| = \frac{1}{v} \cdot v = 1.$!

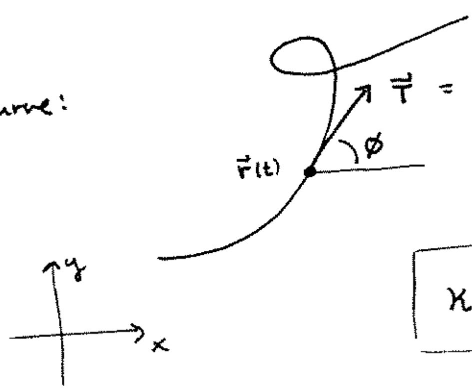
Check that for the circle on page 1 this algorithm reproduces the same unit speed reparameterization

A <u>unit speed parameterization</u> $\vec{\rho}(s)$ is also called an <u>arclength parameterization</u>, since then $\int_0^s |\vec{\rho}\,'(t)|\,dt = \int_0^s 1 = s$

so the length of the curve as t varies from 0 to s exactly equals s.

<u>return to curvature</u>

for a plane curve:



$\vec{T}$ = unit tangent vector at $\vec{r}(t) = \vec{\rho}(s)$ $\left(= \vec{r}(t(s)) \right)$

$$\boxed{\kappa := \left| \frac{d\phi}{ds} \right|}$$

Notice, $\vec{T}(s) = \begin{bmatrix} \cos\phi(s) \\ \sin\phi(s) \end{bmatrix}$

$$\vec{T}\,'(s) = \begin{bmatrix} -\sin\phi \;\; \phi'(s) \\ \cos\phi \;\; \phi'(s) \end{bmatrix}$$

so $\boxed{|\vec{T}\,'(s)| = |\phi'(s)| = \kappa}$

<u>example</u>

$$\vec{\rho}(s) = \begin{bmatrix} 10\cos\frac{s}{10} \\ 10\sin\frac{s}{10} \end{bmatrix}$$

$$\vec{\rho}\,'(s) = \vec{T}(s) = \begin{bmatrix} -\sin\frac{s}{10} \\ \cos\frac{s}{10} \end{bmatrix}$$

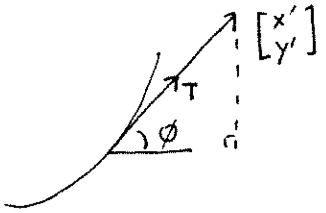
$$\vec{T}\,'(s) = \begin{bmatrix} -\frac{1}{10}\cos\frac{s}{10} \\ -\frac{1}{10}\sin\frac{s}{10} \end{bmatrix}$$

$$|\vec{T}\,'(s)| = \kappa = \frac{1}{10}$$

So, what is the curvature of a radius R circle?

Computing curvature for plane curves:

You don't need to find the explicit arclength reparameterization, you just need to be good at the chain rule



$\vec{F}(t)$ original parameterization, $\vec{F}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

$\phi = \arctan \dfrac{y'(t)}{x'(t)}$ $\qquad \vec{T}(t) = \frac{1}{v}\vec{F}'(t)$, $v = |\vec{F}'(t)| = \sqrt{(x')^2+(y')^2}$

$$\frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} \qquad \frac{ds}{dt} = v \quad \text{so} \quad \frac{dt}{ds} = \frac{1}{v}$$

$$= \frac{1}{v}\frac{d\phi}{dt}$$

$$= \frac{1}{v}\left(\frac{1}{1+\left(\frac{y'}{x'}\right)^2}\right)\underbrace{D_t\left(\frac{y'}{x'}\right)}_{\frac{y''x'-y'x''}{(x')^2}}$$

$$\overset{!}{=} \frac{y''x'-y'x''}{\cancel{v}((x')^2+(y')^2)^{3/2}}$$

so $$\kappa = \frac{|y''x'-y'x''|}{((x')^2+(y')^2)^{3/2}}$$

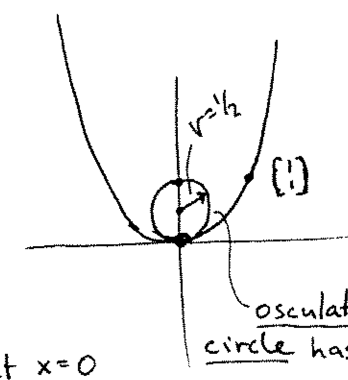
<u>example</u>: Find the curvature of the parabola $y=x^2$, at every point on the parabola.

let $\vec{F}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$

$x = t \qquad x' = 1 \qquad x'' = 0$
$y = t^2 \qquad y' = 2t \qquad y'' = 2$

$$\kappa = \frac{|2\cdot 1 - 2t\cdot 0|}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}$$

so at $\begin{bmatrix} x \\ x^2 \end{bmatrix}$, $\kappa = \dfrac{2}{(1+4x^2)^{3/2}}$



e.g. at $x=0$
$\kappa = 2$

<u>osculating circle</u> has same curvature as curve, is tangent to curve (all at a fixed point), and on the side of the curve which properly reflects the curve's bending.

Space curves $\quad \vec{r} : [a,b] \longrightarrow \mathbb{R}^n \quad$ any $n \geq 2$

We can't use ϕ when $n \neq 2$
but the definition

$$\boxed{\kappa = \left| \frac{d\vec{T}}{ds} \right|}$$

still holds.

Notice, since $\vec{T} \cdot \vec{T} \equiv 1$

$$\frac{d}{ds} \vec{T} \cdot \vec{T} \equiv 0$$

$$2 \frac{d\vec{T}}{ds} \cdot \vec{T} \equiv 0$$

So, if $\frac{d\vec{T}}{ds} \neq 0$ we call $\vec{N} := \frac{1}{\left|\frac{d\vec{T}}{ds}\right|} \frac{d\vec{T}}{ds}$ the unit normal to the curve

$$= \frac{1}{\kappa} \frac{d\vec{T}}{ds}$$

i.e. $\boxed{\frac{d\vec{T}}{ds} = \kappa \vec{N}}$

for plane curves you can get $\vec{N}$ directly from $\vec{T}$

e.g. in parabola example at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Computationally

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \frac{dt}{ds}$$

$$= \frac{1}{v} \frac{d\vec{T}}{dt}$$

$$= \frac{1}{v} \frac{d}{dt}\left(\frac{1}{v} \vec{r}'(t) \right)$$

so $\boxed{\kappa = \frac{1}{v} \left| \frac{d}{dt} \frac{1}{v} \vec{r}'(t) \right|}$

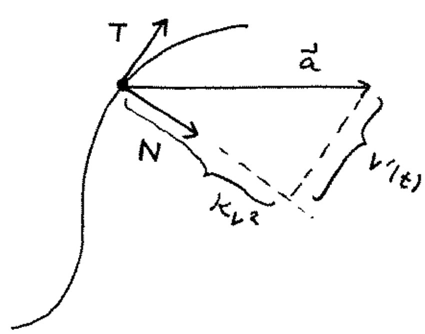
Back to physics:

Normal and tangential components of acceleration

$\vec{T}(t) = \frac{\vec{r}'(t)}{v} \quad$ so

$$\vec{r}'(t) = v \vec{T}(t)$$

$$\vec{r}''(t) = v'(t) \vec{T}(t) + v \underbrace{\frac{d\vec{T}}{dt}}_{\underbrace{\frac{d\vec{T}}{ds}}_{\kappa \vec{N}} \underbrace{\frac{ds}{dt}}_{v}}$$



$$\boxed{\vec{r}''(t) = v'(t) \vec{T} + \kappa v^2 \vec{N}}$$