Math 2210-1
Fri 9/17

This is class #10 – it's time to start the "Calculus" part of our course!

In general we will be considering

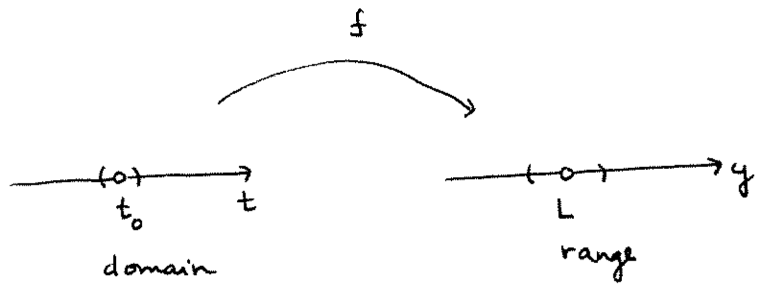
$$\vec{F} : \mathbb{R}^n \to \mathbb{R}^m$$

i.e. domain is subset of $\mathbb{R}^n$
range is subset of $\mathbb{R}^m$
(the input for $\vec{F}$ is a position vector in $\mathbb{R}^n$
the output is a position vector in $\mathbb{R}^m$)

12.1 12.2 Easiest case (most like 1-variable calc) is when domain is 1-dimensional, $n=1$.
We call the domain variable "t" usually, because in applications this variable may actually be "time", and we are modeling <u>particle motion</u>.

Recall
<u>1-variable Calculus</u>

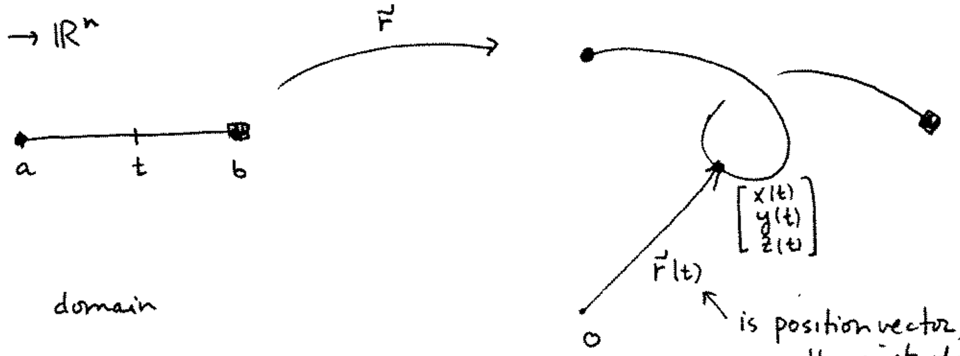
- $\lim_{t \to t_0} f(t) = L$ means?



- f <u>continuous</u> at t_0 means?

- f <u>differentiable</u> at t_0 means?

$\vec{r} : \mathbb{R} \to \mathbb{R}^n$



domain

is position vector, usually just draw its endpoint!

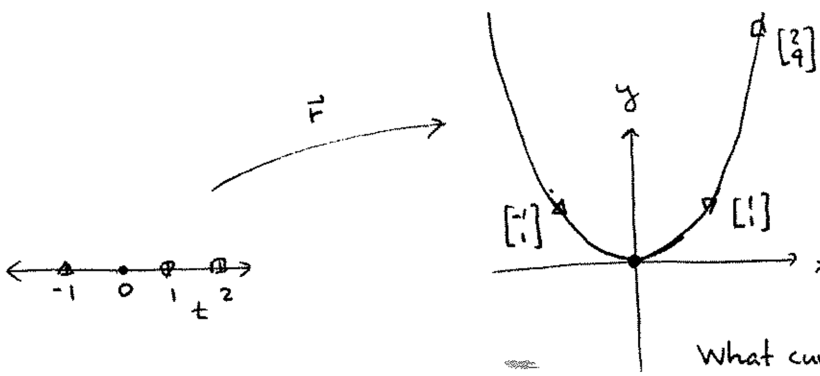
may also write

$x = x(t)$
$y = y(t)$
$z = z(t)$

range.

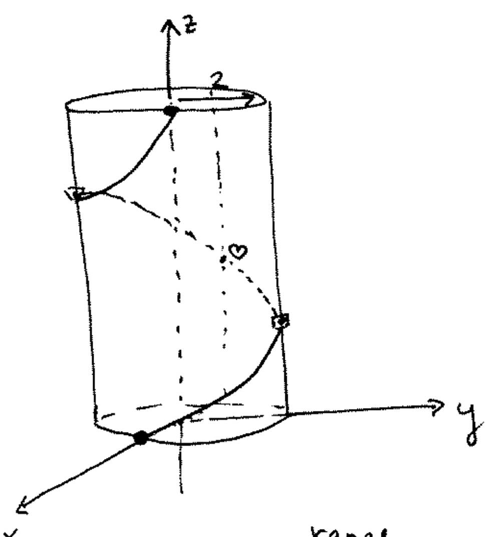
<u>examples</u> ① $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ lines; these are the affine transformations from $\mathbb{R} \to \mathbb{R}^n$

② $\vec{r}(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix}$



What curve do all image (range) points lie on?

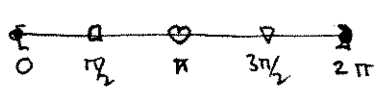
③ $\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \\ 3t \end{bmatrix}$, $0 \le t \le 2\pi$



notice, the image curve lies on the vertical cylinder $x^2+y^2=4$ (radius 2).

range

this is a right-handed helix



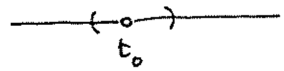
domain

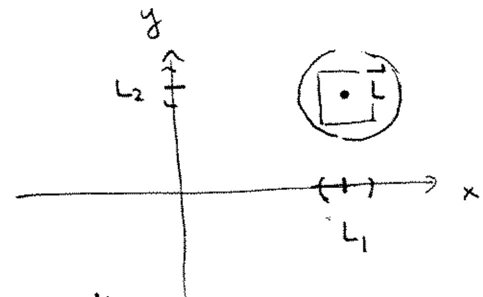
$\vec{r}: \mathbb{R} \to \mathbb{R}^n$

- $\lim_{t \to t_0} \vec{r}(t) = \vec{L}$ means

as t approaches t_0 the distance from $\vec{r}(t)$ to $\vec{L}$ goes to zero

$\left(\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } 0 < |t - t_0| < \delta \Rightarrow |\vec{r}(t) - \vec{L}| < \varepsilon \right)$





- Since the distance from $\vec{r}(t)$ to $\vec{L} = |\vec{r}(t) - \vec{L}|$ is small if and only if each component of $\vec{r}(t)$ is close to the corresponding component of $\vec{L}$, we deduce (for $\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$, $\vec{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$)

$$\lim_{t \to t_0} \vec{r}(t) = \vec{L} \quad \text{iff} \quad \begin{matrix} x(t) \to L_1 \\ y(t) \to L_2 \\ z(t) \to L_3 \end{matrix}$$

<u>example</u>

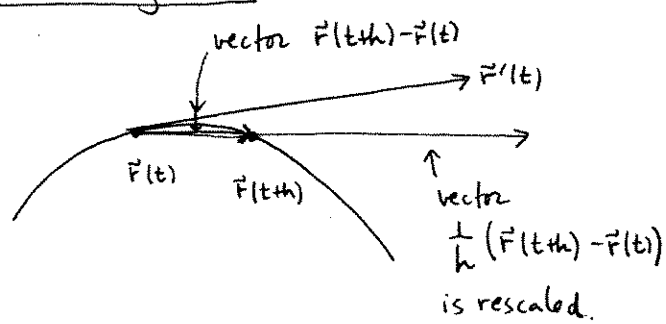
$$\lim_{t \to \pi/2} \begin{bmatrix} 2\cos t \\ 2\sin t \\ 3t \end{bmatrix} = \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$$

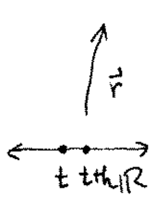
- $\vec{r}(t)$ is <u>continuous</u> at t_0 iff $\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$ "same" as 1-var Calc.

- $\vec{r}(t)$ is <u>diff'ble</u> at t_0 iff $\lim_{t \to t_0} \dfrac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} = \lim_{h \to 0} \dfrac{1}{h}\left(\vec{r}(t_0+h) - \vec{r}(t_0)\right)$

exists. The limit is called $\vec{r}'(t_0)$.

<u>geometric meaning of $\vec{r}'(t)$:</u>





if $\vec{r}'(t)$ exists it must be tangent to the curve.

<u>computing $\vec{r}'(t)$:</u>

$$\lim_{h \to 0} \frac{1}{h}\left(\begin{bmatrix} x(t+h) \\ y(t+h) \\ z(t+h) \end{bmatrix} - \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \right)$$

$$= \lim_{h \to 0} \begin{bmatrix} \frac{x(t+h) - x(t)}{h} \\ \frac{y(t+h) - y(t)}{h} \\ \frac{z(t+h) - z(t)}{h} \end{bmatrix} \quad \left(\text{by vector + \& scalar mult}\right)$$

$$= \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} \quad \text{easy!}$$

For examples 1, 2, 3 on page 2
compute $\vec{r}'(t)$

<u>Example 2</u> $\vec{r}'(1) =$

add this tangent vector
to page 2 drawing.

<u>Example 3</u> $\vec{r}'(\pi/2) =$

add this vector
to page 2 drawing.

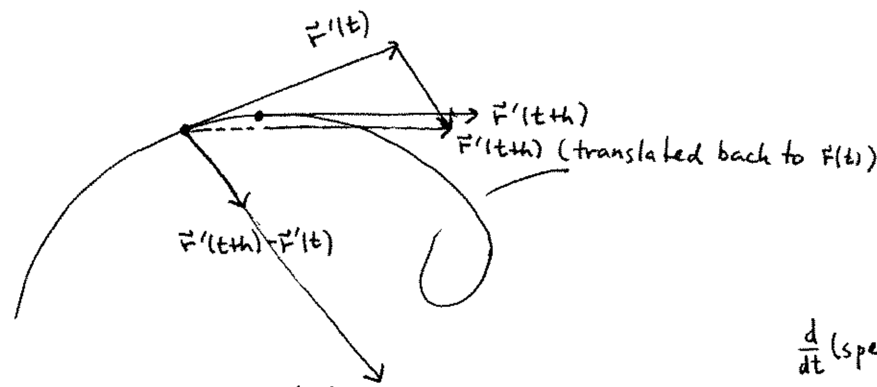
<u>physics</u>: $\vec{r}(t+h) - \vec{r}(t)$ = net displacement

$$\frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \frac{\text{net displacement}}{\text{time elapsed}} = \text{average velocity;} \qquad \left|\frac{\vec{r}(t+h)-\vec{r}(t)}{h}\right| = \text{average speed.}$$

$$\lim_{h\to 0} \frac{1}{h}\left(\vec{r}(t+h)-\vec{r}(t)\right) = \vec{r}'(t) = \vec{v}(t) = \text{velocity.}$$

$$\lim_{h\to 0} \left|\frac{1}{h}\right| \left|\vec{r}(t+h)-\vec{r}(t)\right| = |\vec{r}'(t)| = |\vec{v}(t)| = \text{speed.}$$

$\frac{d}{dt}\vec{r}'(t) = \vec{r}''(t)$ = acceleration vector, measures rate of change of velocity vector:



$\frac{1}{h}\left(\vec{r}'(t+h) - \vec{r}'(t)\right) \approx \vec{r}''(t)$.

$$\frac{d}{dt}(\text{speed}) = \frac{d}{dt}(\vec{v}\cdot\vec{v})^{1/2}$$
$$= \tfrac{1}{2}(\vec{v}\cdot\vec{v})^{-1/2}\left[\vec{v}'\cdot\vec{v} + \vec{v}\cdot\vec{v}'\right]$$
$$= (\vec{v}\cdot\vec{v})^{-1/2}(\vec{v}'\cdot\vec{v})$$

- acceleration vector points in direction curve is bending, and its angle with tangent vector indicates whether particle is speeding up or slowing down!

Example 2 compute $\vec{F}''(1)$
add to picture

Example 3 : compute $\vec{F}''(\pi/2)$
add to picture

Derivative rules (easy to check)

✓ 1. $D_t\left[\vec{u}(t)+\vec{v}(t)\right] = \vec{u}'(t)+\vec{v}'(t)$

✓ 2. $D_t\left[c\vec{u}(t)\right] = c\,\vec{u}'(t)$

3. $D_t\left[h(t)\vec{u}(t)\right] = h'\vec{u}+h\vec{u}'$

4. $D_t\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}'\cdot\vec{v}+\vec{u}\cdot\vec{v}'$

5. $D_t\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}'\times\vec{v}+\vec{u}\times\vec{v}'$

universal product rule
can check each one individually, but the idea is below

Let $A(t)$, $B(t)$ be scalar, vector, or matrix fcns, and let
$*$ be some sort of product that distributes over addition, and "respects" scalar mult.
Then $D_t\left(A(t)*B(t)\right) = A'(t)*B + A*B'$:

$$\lim_{h\to 0}\frac{1}{h}\left[A(t+h)*B(t+h)-A(t)*B(t)\right] = \lim_{h\to 0}\frac{1}{h}\left[A(t+h)*\left(B(t+h)-B(t)\right) + \left(A(t+h)-A(t)\right)*B(t)\right]$$

$$= \lim_{h\to 0} A(t+h)*\left[\frac{B(t+h)-B(t)}{h}\right] + \left[\frac{A(t+h)-A(t)}{h}\right]*B(t)$$

$$= A*B' + A'*B \quad !$$