Math 2210-1
Monday 13 Sept.

Matrix geometry & algebra cont'd from Friday

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ is an affine map (function) iff

$$\vec{F}(\vec{x}) = A\vec{x} + \vec{b}$$

(this generalizes $f(x) = mx + b$, $f: \mathbb{R} \to \mathbb{R}$)

i.e.

$$\vec{F}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & & a_{2n} \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$:= \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$= \begin{bmatrix} Row_1(A)\cdot\vec{x} \\ Row_2(A)\cdot\vec{x} \\ \vdots \\ Row_m(A)\cdot\vec{x} \end{bmatrix}$$

You can think of $\vec{F}(\vec{x})$ as the composition of first multiplying $A\vec{x}$, and then translating by $\vec{b}$. In case $\vec{b} = \vec{0}$, (no translation), $\boxed{\vec{F}(\vec{x}) = A\vec{x} \text{ is called } \underline{\text{linear}}}$

By properties of dot product (or check directly),

$$\boxed{\begin{array}{ll} \text{(i)} & A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w} \\ \text{(ii)} & A(t\vec{v}) = tA\vec{v}, \quad t \in \mathbb{R} \end{array}}$$

<u>We deduce</u>: Let $\vec{F}(\vec{x}) = A\vec{x} + \vec{b}$

- Let L be a line in domain $\mathbb{R}^n$. Then the image of L is a line in $\mathbb{R}^m$ (or a point.) Parallel lines are mapped to parallel lines

Reason: Let L_1: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$ (parametrize)
Then $\vec{F}(\vec{r}(t)) = A(\vec{r}_0 + t\vec{v}) + \vec{b}$
$= (A\vec{r}_0 + \vec{b}) + t\,A\vec{v}$

If $A\vec{v} \neq \vec{0}$ this is line; else point

If $L_2 \parallel L_1$ then can use same direction vector $\vec{v}$ to parameterize it, so image line also has direction vector $A\vec{v}$ ∎

- Coordinate grids in $\mathbb{R}^n$ are mapped to parallelogram (or parallelepiped) grids in $\mathbb{R}^m$

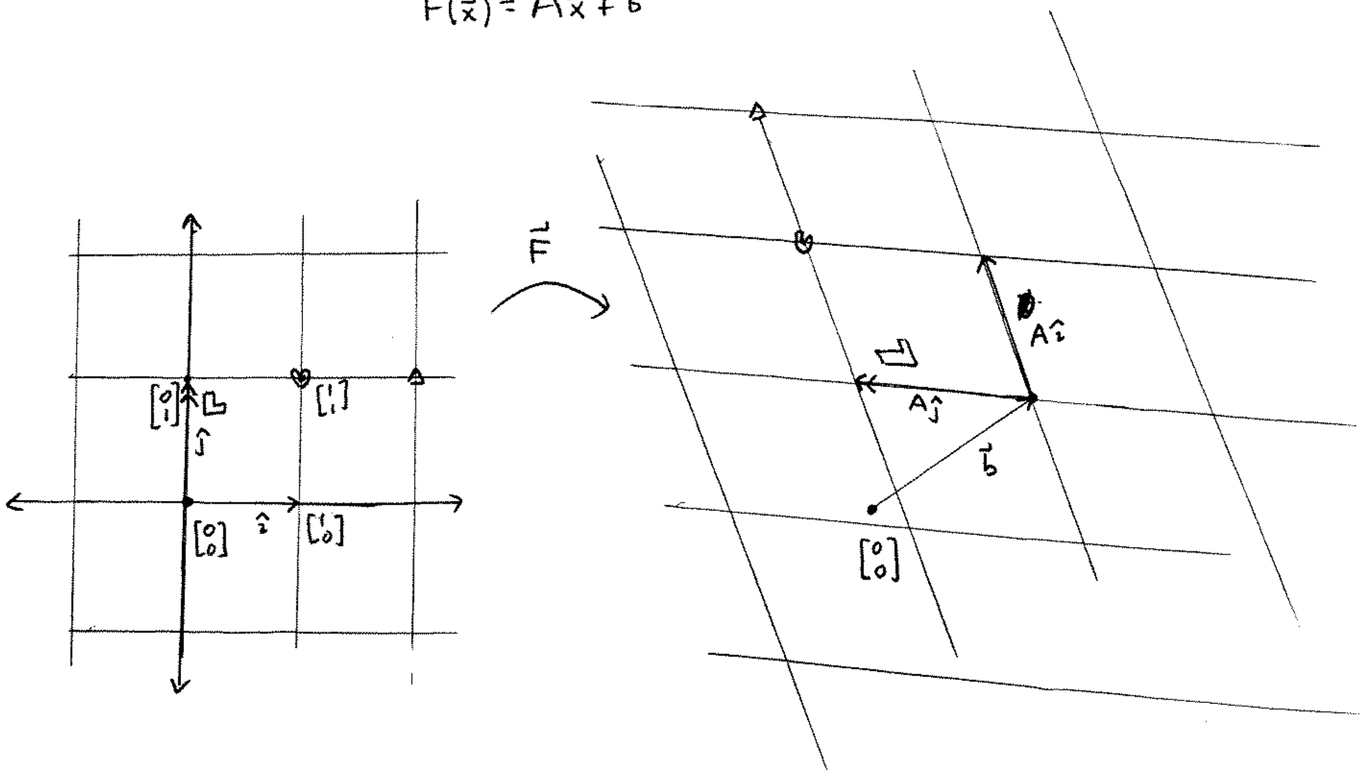
e.g. if $n=2$, $F(s\hat{\imath} + t\hat{\jmath}) = A(s\hat{\imath} + t\hat{\jmath}) + \vec{b}$
$= s(A\hat{\imath}) + t(A\hat{\jmath}) + \vec{b}$

So, e.g. $F(3\hat{\imath} + 4\hat{\jmath}) = 3(A\hat{\imath}) + 4(A\hat{\jmath}) + \vec{b}$

etc. ∎

Schematic for $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\vec{F}(\vec{x}) = A\vec{x} + \vec{b}$$



Note also if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$A\hat{\imath} = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = col_1(A)$$

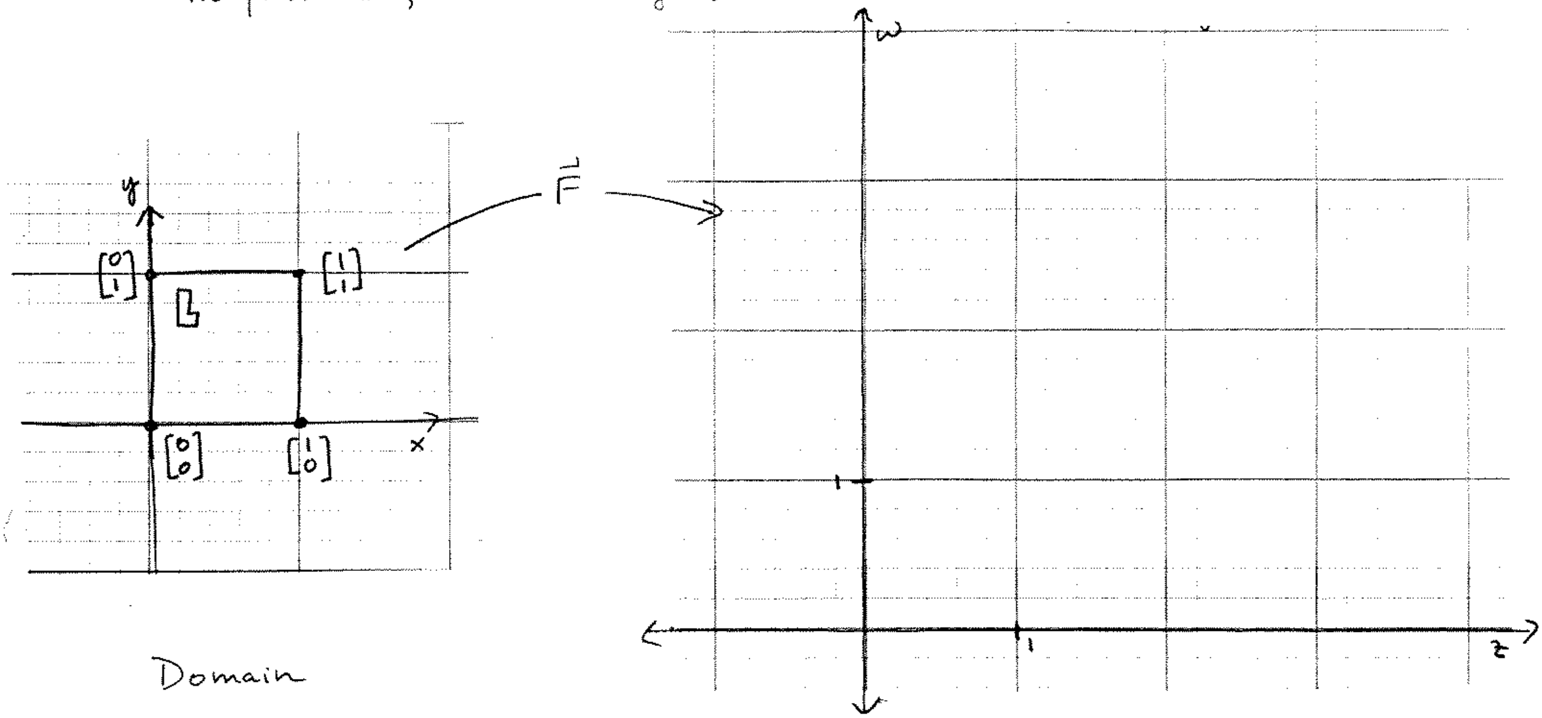
$$A\hat{\jmath} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = col_2(A)$$

Examples : We call the unit square in $\mathbb{R}^2$ the "L-box" after we put an "L" into the upper left corner.

1. Draw a picture which shows the geometry of the affine map

$$\vec{F}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

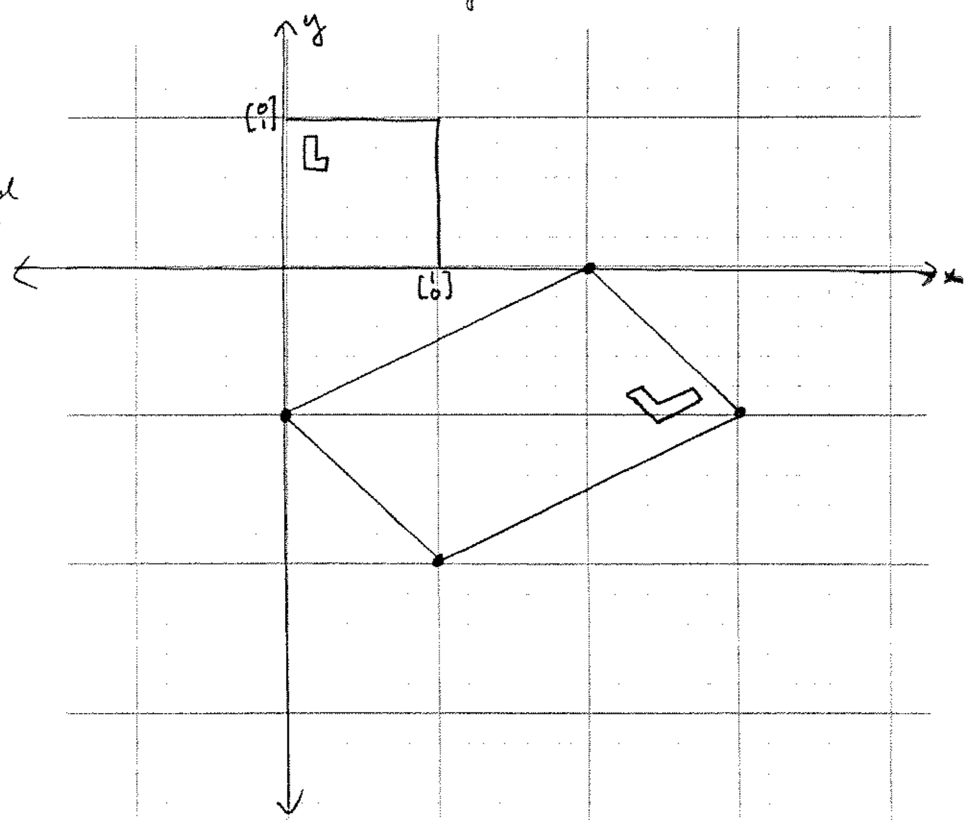
In particular, show the image of the L box



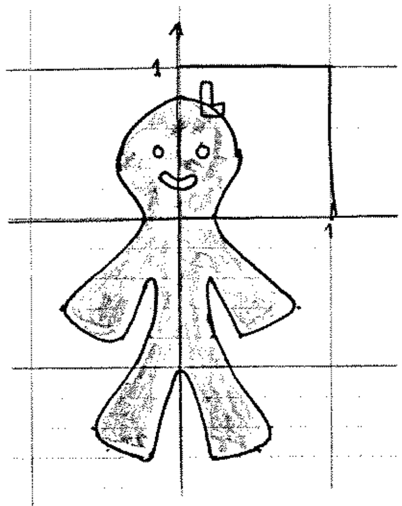
Domain

Range

2. Reconstruct the affine transformation from the L-box picture. In this case (like in HW), I've plotted the range picture, and copied the original L-box onto it for reference.



3. Recall gingerBob from the homework assignment:

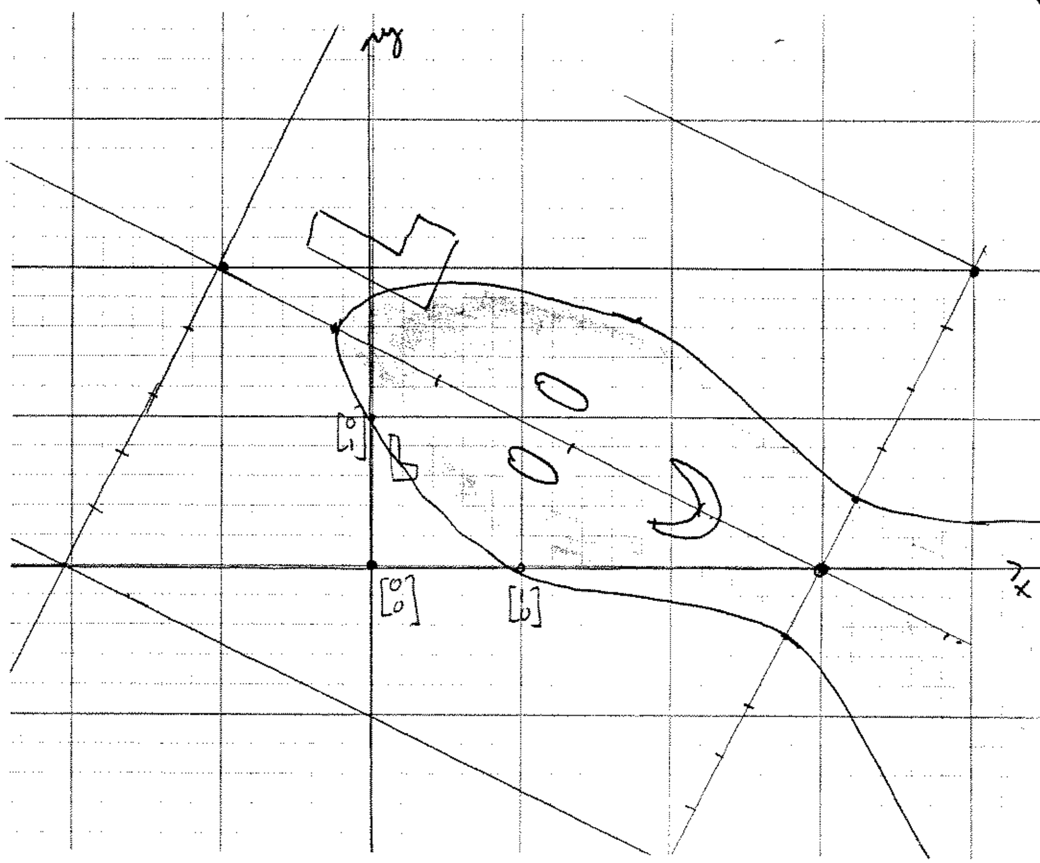


By counting squares, gingerBob has area $A \approx 1.76$ square units.

What is transformed gingerBob's area, if he is transformed by $\vec{F}$ from 1 or 2?

e.g., he looks something like this if transformed by $F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & -4 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}$:

these columns are now correct for this page, but switched from 1 on page 2. (In the switched example Bob would be turned over and his axis would point in the $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ direction!)



Summary of Bob's conclusions

- the determinant of a (square) matrix stays the same if you interchange rows & cols
- So, if $\vec{f}(\vec{x}) = A\vec{x} + \vec{b}$, $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$

  $|\det(A)|$ is volume expansion factor

  and the sign of $\det A$ tells you whether you kept or switched orientation.

Matrix algebra (and what it means)

- Add and scalar multiply matrices just like vectors – entry by entry (of course, matrices must be same size)

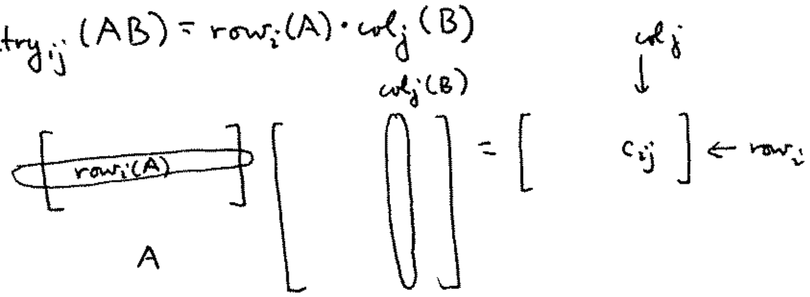
$$\text{entry}_{ij}(A+B) = a_{ij} + b_{ij}$$
$$\text{entry}_{ij}(tA) = t\,a_{ij}$$

$$3\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & & \\ & & \end{bmatrix}$$

- If $A_{m\times k}$, $B_{k\times n}$

  then $C_{m\times n} = AB$ is defined

  by $\text{entry}_{ij}(AB) = \text{row}_i(A) \cdot \text{col}_j(B)$



e.g. compute AB and BA for

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$(A_{2\times \cancel{3}})(B_{\cancel{3}\times 2}) = (AB)_{2\times 2}$$
$$(B_{3\times \cancel{2}})(A_{\cancel{2}\times 3}) = (BA)_{3\times 3}$$

Matrix multiplication corresponds to composition of linear fcns (and that's why it's defined as it is!):

For $A_{k\times n}$ consider $\vec{F}(\vec{x}) = A\vec{x}$

$\vec{F}: \mathbb{R}^n \to \mathbb{R}^k$

then for $B_{m\times k}$ consider $\vec{G}(\vec{y}) = B\vec{y}$, $\quad \vec{G}: \mathbb{R}^k \to \mathbb{R}^m$

then $\vec{G}\circ\vec{F} : \mathbb{R}^n \to \mathbb{R}^k \to \mathbb{R}^m$ (the composition)

and $\vec{G}\circ\vec{F}$ is linear (because the components of ans will be sums of multiples of the x_i)

Let C be the matrix of $\vec{G}\circ\vec{F}$

then
$$
\begin{aligned}
col_j(C) &= \vec{G}\left(\vec{F}\left(\begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} & \leftarrow j^{th}\text{spot} \end{array}\right)\right) \\
&= \vec{G}\left(col_j(A)\right) \\
&= B\, col_j(A) \\
&= col_j(BA) \qquad \longrightarrow \text{so } C = BA\,!
\end{aligned}
$$

Matrix algebra

$$A+B = B+A$$
$$A+(B+C) = (A+B)+C$$
$$A(BC) = (AB)C$$
$$A(B+C) = AB+AC$$
$$(A+B)C = AC+BC$$

but

$AB \neq BA$ in general.

how many of these could you verify?