Math 2210
Wed 9/1

HW (complete assignment on Friday)
§11.3 1, 7, 8, 11, 15 17 18 23 25 36

- Remark on physics "work" $\sim \overrightarrow{PQ}$
  for straight-line displacement in a constant force field $\vec{F}$

$$W := (\text{comp}_{\overrightarrow{PQ}} \vec{F})(\text{distance})$$
$$= \left(\vec{F} \cdot \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\right)(|\overrightarrow{PQ}|)$$
$$= \vec{F} \cdot \overrightarrow{PQ}$$

e.g. in HW you handed in today.

## Determinants and cross products

2×2:

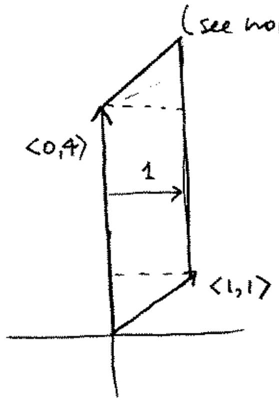
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := a_{11}a_{22} - a_{12}a_{21}$$

geometric meaning! (see notes & HW)

$\vec{u} = \langle u_1, u_2 \rangle$
$\vec{v} = \langle v_1, v_2 \rangle$

$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ = signed area of parallelogram spanned by $\vec{u}$ & $\vec{v}$!

$a_{ij}$: $i$ = row, $j$ = column




Area = bh = 4·1 = 4
from geometry

examples

$\begin{vmatrix} 1 & 1 \\ 0 & 4 \end{vmatrix} = +4$ "right-handed"

$\begin{vmatrix} 0 & 4 \\ 1 & 1 \end{vmatrix} = -4$ "left-handed"

signed area from determinants!

3×3:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

cross out row 1, col 1 — delete row 1 col 2 — delete row 1 col 3

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

**Theorem**: You can compute det by expanding across <u>any</u> row or down <u>any</u> column, using sub-2×2 dets, and a checkerboard $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ of +,−'s.
(see HW)

Cross product for 3-vectors $\vec{u}, \vec{v}$:

$$\vec{u}\times\vec{v} := \text{"}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\text{"} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

$$= \langle u_2v_3 - v_2u_3,\ v_1u_3 - u_1v_3,\ u_1v_2 - v_1u_2 \rangle$$

Cross product algebra:

✓ 1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$ ! not commutative

✓ 2. $(t\vec{u})\times\vec{v} = \vec{u}\times(t\vec{v}) = t(\vec{u}\times\vec{v})$

✓ 3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

?✓ 4. $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$

?? 5. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$ ! not associative

Cross product geometry:

can prove (4.) from this

observation 1: $\vec{w}\cdot(\vec{u}\times\vec{v}) = w_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

deduce

$\vec{u}\cdot(\vec{u}\times\vec{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0!$ (e.g. expand across bottom row!)
(2 rows equal $\Rightarrow$ det $= 0$)

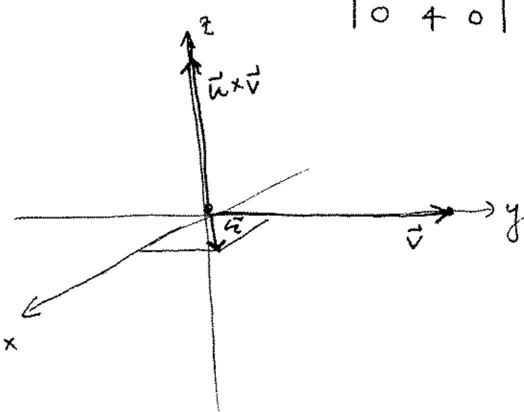
$\vec{v}\cdot(\vec{u}\times\vec{v}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0!$

$\boxed{\text{So } \vec{u}\times\vec{v} \text{ is perpendicular to } \vec{u} \,\&\, \vec{v}}$

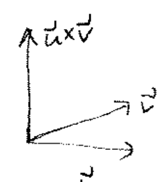
In fact, $\{\vec{u}, \vec{v}, \vec{u}\times\vec{v}\}$ is always right-handed

Example (see page 1)

$$\langle 1,1,0\rangle \times \langle 0,4,0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 4 & 0 \end{vmatrix} = \langle 0,0,4\rangle$$




in general



How big is $|\vec{u}\times\vec{v}|$?

<u>Algebra lemma</u> (see homework)

$$|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2$$

Thus

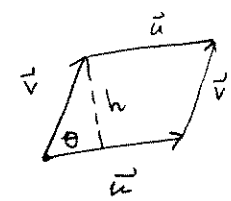
$$|\vec{u}\times\vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2\left(1-\cos^2\theta\right), \quad \left(\text{since } \vec{u}\cdot\vec{v} = |u||v|\cos\theta\right)$$
$$= |\vec{u}|^2|\vec{v}|^2\sin^2\theta$$

<u>Geometric summary</u>

(1) $\vec{u}\times\vec{v} \perp \vec{u},\vec{v}$ (<u>and</u> right-hand rule)

(2) $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

= area of parallelogram determined by $\vec{u},\vec{v}$

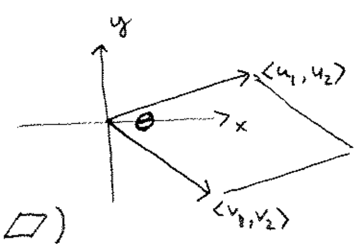



Area $= bh$
$= |\vec{u}|\,|\vec{v}|\sin\theta$
$= |\vec{u}\times\vec{v}|$

<u>Example</u> $\vec{u} = \langle u_1, u_2, 0\rangle$
$\vec{v} = \langle v_1, v_2, 0\rangle$

i.e. $\vec{u},\vec{v}$ are in (x-y) plane

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \left\langle 0, 0, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right\rangle$$




so area of ▱
$= |u_1v_2 - v_1u_2|$
$= \left|\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right|$

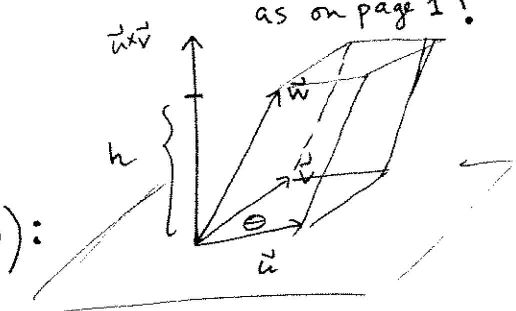
Can you explain why the det (without abs value) gives <u>signed</u> area? as on page 1?

If $2\times 2$ dets give signed area, (of ▱) what do $3\times 3$ dets yield?

<u>Theorem</u>:

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \pm \text{ Volume of parallelepiped determined by } \{\vec{u},\vec{v},\vec{w}\}$$

(+ if $\{\vec{u},\vec{v},\vec{w}\}$ right-handed, else −)




Vol = (area of base)(height)
$= \left(|\vec{u}||\vec{v}|\sin\theta\right)\left(|\operatorname{comp}_{\vec{u}\times\vec{v}}\vec{w}|\right)$

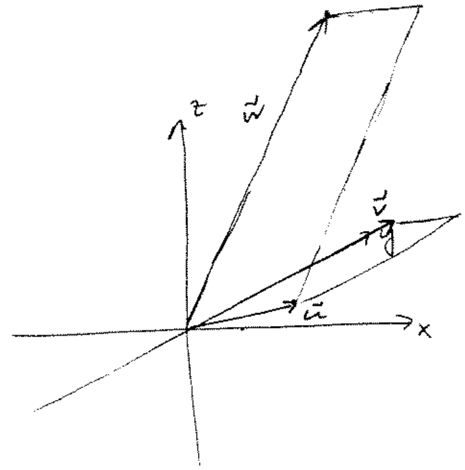
‖ (4) prop

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \vec{w}\cdot(\vec{u}\times\vec{v}) = \underbrace{|\vec{u}\times\vec{v}|}_{|\vec{u}||\vec{v}|\sin\theta}\underbrace{\left(\vec{w}\cdot\frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|}\right)}_{\operatorname{comp}_{\vec{u}\times\vec{v}}\vec{w}}$$

compare! ∎

Example

find volume of parallelepiped determined by $\langle 1,1,0\rangle$
$\langle 0,4,0\rangle$
$\langle 1,2,6\rangle$




fill in picture!

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 4 & 0 \\ 1 & 2 & 6 \end{vmatrix} = 1(24) - 1(0) + 0(-4)$$

$$= 24$$

Example find volume of pyramid with $\Delta$ base (tetrahedron!) with ~~co~~ vertices

$(0,0,0)$
$(1,1,0)$
$(0,4,0)$
$(1,2,6)$

$V = \frac{1}{6}$ vol of parallelepiped! $= 4$!