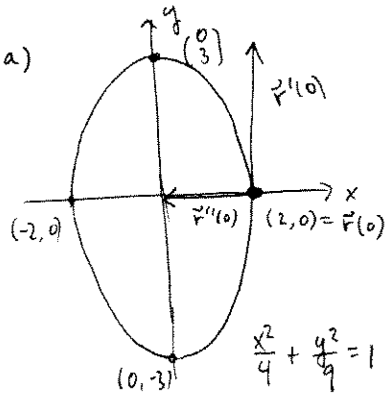
Math 2210-1
Sample final exam sol'ns
12/2004

1a)



1b) if $x=2\cos t$
$y=3\sin t$

then $\frac{x^2}{4}+\frac{y^2}{9}=\frac{\cancel{4}\cos^2 t}{\cancel{4}}+\frac{\cancel{9}\sin^2 t}{\cancel{9}}=1$

so $\vec{r}(t)$ satisfies the implicit ellipse equation

1c) $\vec{r}(t)=\begin{bmatrix}2\cos t\\3\sin t\end{bmatrix}$, $\vec{r}'(t)=\begin{bmatrix}-2\sin t\\3\cos t\end{bmatrix}$, $\vec{r}''(t)=\begin{bmatrix}-2\cos t\\-3\sin t\end{bmatrix}$

$\vec{r}(0)=\begin{bmatrix}2\\0\end{bmatrix}$, $\vec{r}'(0)=\begin{bmatrix}0\\3\end{bmatrix}$ $\vec{r}''(0)=\begin{bmatrix}-2\\0\end{bmatrix}$

1d) $\frac{d}{dt}v^2=\frac{d}{dt}\vec{r}'(t)\cdot\vec{r}'(t)=2\vec{r}''(t)\cdot\vec{r}'(t)$

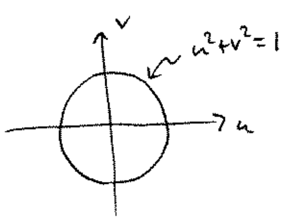
@ $t=0$, $=2\begin{bmatrix}-2\\0\end{bmatrix}\cdot\begin{bmatrix}0\\3\end{bmatrix}=0$

so particle is neither speeding up or slowing down at this instant

2a) $u=x/2$ $\qquad x=2u$
$v=y/3$ $\qquad y=3v$

then $\frac{x^2}{4}+\frac{y^2}{9}=1$ is
equivalent to $u^2+v^2=1$

(and $\frac{x^2}{4}+\frac{y^2}{9}\le 1$ same as $u^2+v^2\le 1$)



Thus $X(u,v)=\binom{2u}{3v}$ maps the unit disk to the inside of the (a) ellipse.

so, by COV
$$\text{Area}=\iint_{\frac{x^2}{4}+\frac{y^2}{9}\le 1} 1\,dxdy=\iint_{\substack{u\text{-}v\\ \text{region}}} 1\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv=\iint_{\text{unit disk}} 6\,dudv=6\pi$$

$X'(u,v)=\begin{bmatrix}2&0\\0&3\end{bmatrix}$

$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=6$

2b) $\oint_{\partial R} P\,dx+Q\,dy=\iint_R Q_x-P_y\,dxdy$ $\qquad \langle P,Q\rangle=\langle -y,x\rangle$

$=\iint_R \underbrace{1--1}_{2}\,dxdy=2\cdot\text{area of ellipse}$

so $A=6\pi$

$\vec{r}(t)=\binom{2\cos t}{3\sin t}$

$$\int_0^{2\pi}\langle \underset{P}{-3\sin t},\underset{Q}{2\cos t}\rangle\cdot\underbrace{\langle -2\sin t,3\cos t\rangle}_{\vec{r}'(t)}\,dt=\int_0^{2\pi}6\,dt=12\pi$$

$6(\sin^2 t+\cos^2 t)$
$=6$

3 a) $\int_0^4 \int_{y/2}^{y^{1/2}} 4x\,dx\,dy = \int_0^4 2x^2\Big]_{y/2}^{y^{1/2}} dy = \int_0^4 2y - 2\frac{y^2}{4}\,dy = y^2 - \frac{1}{6}y^3\Big]_0^4 = 16 - \underbrace{\frac{64}{6}}_{32/3}$

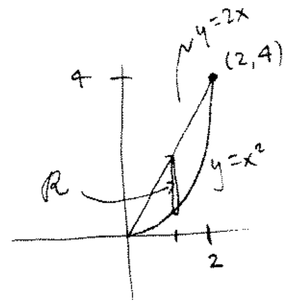
$= \frac{48-32}{3} = \frac{16}{3}$

b) $0 \le y \le 4$

for fixed y, $\underbrace{y/2} \le x \le \underbrace{y^{1/2}}$

$x = y/2$ $\quad$ $x = y^{1/2}$

$y = 2x$ $\quad$ lies on $y = x^2$



c), using sketch of region, 3a) integral also equals

$$\int_0^2 \underbrace{\int_{x^2}^{2x} 4x\,dy}_{4x\,y\big]_{x^2}^{2x}}\,dx = \int_0^2 8x^2 - 4x^3\,dx = \frac{8}{3}x^3 - x^4\Big]_0^2 = \frac{64}{3} - 16 = \frac{64-48}{3} = \frac{16}{3}$$

4. Check cross-partial condition $Q_x = P_y$

(a) $\vec{F} = \langle \sin x + e^x \cos y,\ 3y^2 - e^x \sin y \rangle$

$\left.\begin{array}{l} P_y = -e^x \sin y \\ Q_x = -e^x \sin y \end{array}\right\}$ so $\vec{F} = \nabla f$

(b) $\vec{F} = \langle y, -x \rangle$

$\left.\begin{array}{l} Q_x = -1 \\ P_y = 1 \end{array}\right\}$ $\neq$, so $\vec{F} \neq \nabla f$

$f_x = \sin x + e^x \cos y$

$f = \int \sin x + e^x \cos y\,dx = -\cos x + e^x \cos y + g(y)$

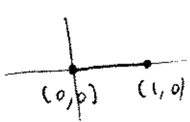
so $f_y = \cancel{-e^x \sin y} + g'(y) = Q = 3y^2 \cancel{- e^x \sin y}$

$g'(y) = 3y^2$

$g(y) = y^3 + C$

So $f(x,y) = -\cos x + e^x \cos y + y^3 + C$

5. a) $\int_C \vec{F} \cdot d\vec{x}$ $\quad$ using all-purpose def: $\vec{r}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$ $\quad 0 \le t \le 1$, $\vec{r}\,'(t) = \langle 1, 0 \rangle$



$\int_0^1 \langle \sin t + e^t \overset{1}{\cos 0},\ 0 - e^t \sin 0 \rangle \cdot \langle 1, 0 \rangle\,dt$

$= \int_0^1 \sin t + e^t\,dt = -\cos t + e^t\Big]_0^1 = -\cos 1 + e - (-1+1)$

$= e - \cos 1$

5b) $\int_C \vec{F} \cdot d\vec{x} = f(B) - f(A)$

$= f(1,0) - f(0,0)$

$= -\cos 1 + e - 0$

$= e - \cos 1$

(or, using different language, $\int_C P\,dx + Q\,\underset{0}{dy} = \int_0^1 \sin x + e^x \cos 0\,dx$

6 a) $F(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$

$$F'(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

6b) $\det(F'(r,\theta)) = r\cos^2\theta + r\sin^2\theta = r$

so $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = |r| = r$ is the area expansion factor in the general C.O.V. formula,

i.e. in polar coords,

$$dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \, dr\,d\theta = r\,dr\,d\theta$$

6c) $x = r\cos\theta$ $\qquad$ $y = r\sin\theta$

$dx = x_r\,dr + x_\theta\,d\theta$ $\qquad$ $dy = y_r\,dr + y_\theta\,d\theta$

$= \cos\theta\,dr - r\sin\theta\,d\theta$ $\qquad$ $= \sin\theta\,dr + r\cos\theta\,d\theta$

@ $(r,\theta) = (2,0)$ $\qquad$ $(x,y) = (2\cos 0, 2\sin 0) = (2,0)$

$dr = .1$

$d\theta = -.1$

so $\left.\begin{aligned} dx &= 1(+.1) + 0, &&= +.1 \\ dy &= 0\,dr + 2(-.1) &&= -.2 \end{aligned}\right\}$ $\boxed{\begin{aligned} x + dx &= 2.1 \\ y + dy &= -.2 \end{aligned}}$

same.

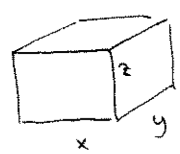
same computation with matrices:

F' matrix

$$\vec{F}(r+dr, \theta+d\theta) \approx \vec{F}(r,\theta) + \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$\vec{F}(2+.1, 0-.1) \approx \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} .1 \\ -.1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} .1 \\ -.2 \end{bmatrix} = \begin{bmatrix} 2.1 \\ -.2 \end{bmatrix}$$

7.



$V = xyz = 60$

$C = \underbrace{2xz + 2yz}_{\text{sides}} + \underbrace{3 \cdot xy}_{\text{bottom}} + \underbrace{2 \cdot xy}_{\text{top}} = 2xz + 2yz + 5xy$

(a) $z = \frac{60}{xy}$ $\qquad$ $C = 2x\frac{60}{xy} + 2y\frac{60}{xy} + 5xy = \frac{120}{y} + \frac{120}{x} + 5xy$

$C_x = 0 = -\frac{120}{x^2} + 5y$ $\qquad$ $5x^2y = 120$ $\qquad$ $x^2y = 24$

$C_y = 0 = -\frac{120}{y^2} + 5x$ $\qquad$ $5xy^2 = 120$ $\qquad$ $xy^2 = 24$

$\frac{x}{y} = \frac{x^2y}{xy^2} = \frac{24}{24} = 1$

$\Rightarrow x = y$

$\Rightarrow x = y = \sqrt[3]{24}$

$= 2 \cdot 3^{1/3}$
$= x = y$

$\Rightarrow z = \frac{60}{4 \cdot 3^{2/3}}$

$= \frac{5 \cdot 3}{3^{2/3}} = 5 \cdot 3^{1/3}$

7(b) $\nabla C = \lambda \nabla V$

$\langle 2z + 5y, 2z + 5x, 2x + 2y \rangle = \lambda \langle yz, xz, xy \rangle$

$2z + 5y = \lambda yz$ $\qquad$ $\lambda = \frac{2z+5y}{yz} = \frac{2}{y} + \frac{5}{z}$

$2z + 5x = \lambda xz$ $\qquad$ $\lambda = \frac{2z+5x}{xz} = \frac{2}{x} + \frac{5}{z}$ $\Big\}$ $x = y$ $\Downarrow$

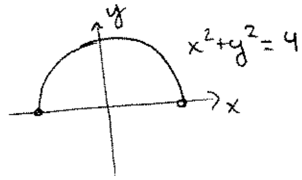
$2x + 2y = \lambda xy$ $\qquad$ $\lambda = \frac{2x+2y}{xy} = \frac{2}{y} + \frac{2}{x} = \frac{4}{x}$

$\Big\}$ $\frac{2}{x} + \frac{5}{z} = \frac{4}{x}$

$\frac{5}{z} = \frac{2}{x}$

$\frac{z}{5} = \frac{x}{2} \Rightarrow \boxed{z = \frac{5}{2}x}$

so $60 = x^2\left(\frac{5}{2}x\right) \Rightarrow x^3 = 60 \cdot \frac{2}{5} = 24 \Rightarrow x = y = 2 \cdot 3^{1/3}$, $z = 5 \cdot 3^{1/3}$

8. 

a) mass = length · density , since density is const

$= \frac{1}{2}(2\pi \cdot 2)\cdot 200 = 400\pi$ gm

b) $\boxed{\bar{x}=0}$ by symmetry

$$\bar{y} = \frac{1}{m}\int_C y\,\delta\, ds = \frac{1}{400\pi}\cdot \overset{\delta}{\overset{\downarrow}{200}} \underbrace{\int_C y\,ds}$$

$$\vec{r}(t) = \begin{bmatrix} 2\cos t \\ 2\sin t \end{bmatrix}$$

$$\vec{r}\,'(t) = \begin{bmatrix} -2\sin t \\ 2\cos t \end{bmatrix}$$

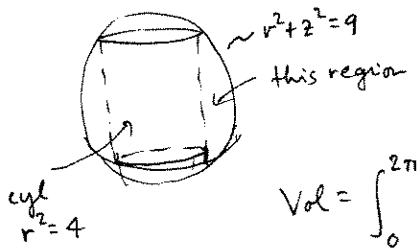
$$ds = |\vec{r}\,'(t)|\,dt = 2\,dt$$

$$\int_0^\pi \underset{y}{(2\sin t)}\ \underset{ds}{2\,dt}$$

$$= 4\int_0^\pi \sin t\, dt = 8$$

so $\bar{y} = \frac{1}{2\pi}\cdot 8 = \boxed{\frac{4}{\pi}}$

9. Cylindrical coords



$$\text{Vol} = \int_0^{2\pi}\int_2^3 \underbrace{\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\,dz}_{2r\sqrt{9-r^2}\,dr}\,dr\,d\theta \qquad (dV = r\,dz\,dr\,d\theta)$$

$$-\frac{2}{3}(9-r^2)^{3/2}\Big]_2^3$$

$$\int_0^{2\pi} \frac{2}{3}\,5^{3/2}\,d\theta = \boxed{\frac{4\pi}{3}\,5^{3/2}}$$

10. $\vec{F} = \langle e^x\cos y,\ e^x\sin y + z,\ xy\rangle$

$\vec{\nabla}\cdot\vec{F} = \langle \partial_x, \partial_y, \partial_z\rangle \cdot \vec{F}$

$= e^x\cos y + e^x \sin y + 0$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ e^x\cos y & e^x\sin y + z & xy \end{vmatrix} = \langle \underset{-1}{x},\ -y,\ e^x\sin y - -e^x\sin y\rangle = \langle x-1, -y, 2e^x\sin y\rangle$$