Name..

I.D. number..

Math 2210-1

Practice Exam 1

September 27, 2004

This exam is closed-book and closed-note. You may use a scientific calculator, but not one which is capable of graphing or of solving linear algebra equations. **In order to receive full or partial credit on any problem, you must show all of your work** and **justify your conclusions.** There are 100 points possible. The point values for each problem are indicated in the right-hand margin. (On the actual exam there will be space for you to do your work. On this practice exam use your own paper.) **Good Luck!**

1a) Consider the triangle with vertices the origin 0=(0,0,0), and the points P=(1,2,1) and Q=(0,1,-1). Find the area of this triangle.

(10 points)

1b) Find an (implicit) equation of the form ax+by+cz=d for the plane containing the three points in (1a).

(5 points)

1c) Express this same plane parametrically.

(5 points)

1d) Consider the line through R=(1,1,0) and S=(0,1,1). Express this line parametrically.

(5 points)

1e) Find the intersection of the line in (1d) with the plane in (1a).

(10 points)

1f) Consider a second plane, given implicitly by

$$x-y-z=2$$

Find the intersection set of this plane with the plane in (1a).

(10 points)

1a) Area $= \frac{1}{2}|\vec{OP}\times\vec{OQ}|$; $\begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \langle -3, 1, 1\rangle$ so area $= \boxed{\frac{1}{2}\sqrt{11}}$

1b) from 1a), normal vector to plane is $\langle -3,1,1\rangle$
so eqtn is $-3x+y+z=d$. plugging $\vec{0}$ into eqtn deduced $=0$
$\boxed{-3x+y+z=0}$

1c) $z=t$
$y=s$
$x=\frac{-s-t}{-3}=\frac{1}{3}(s+t)$
or, in vector form $\boxed{\begin{bmatrix}x\\y\\z\end{bmatrix}=s\begin{bmatrix}1/3\\1\\0\end{bmatrix}+t\begin{bmatrix}1/3\\0\\1\end{bmatrix}}$

1d) $\vec{RS}=\langle -1,0,1\rangle$ so, e.g. $\boxed{\vec{r}(t)=\begin{bmatrix}1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}}$

1e) when does $\vec{r}(t)$ satisfy 1b) eqtn? $-3(1-t)+1+t=0 \Rightarrow$ $4t=2$; $t=\frac{1}{2}$
$(-3x+y+z=0)$ so $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\boxed{\begin{bmatrix}1/2\\1\\1/2\end{bmatrix}}$

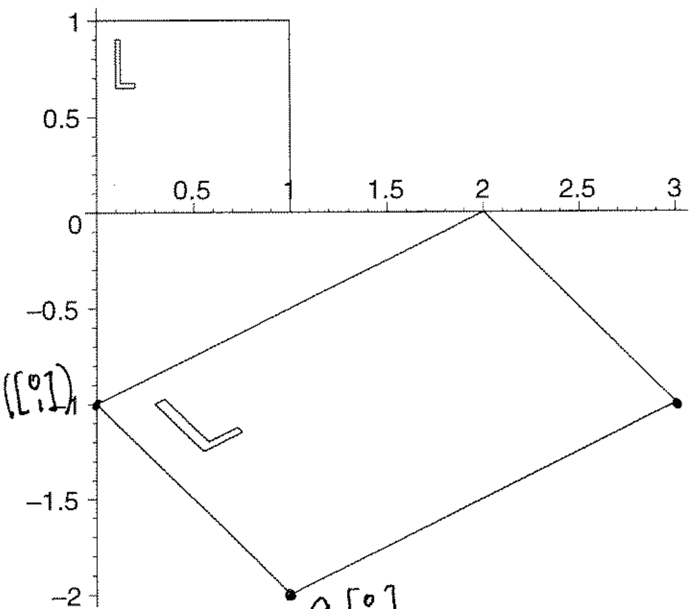
1f):
$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ -3 & 1 & 1 & 0 \end{array}\right]$$

$3R_1+R_2$
$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & -2 & -2 & 6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 1 & -3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

$z=t$
$y=-3-t$
$x=-1$

$\boxed{\text{line } \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\-3\\0\end{bmatrix}+t\begin{bmatrix}0\\-1\\1\end{bmatrix}}$

2) Consider an affine map from R^2 to R^2. Consider the following L-box diagram, which shows the unit "L-box" and where it is transformed to by our mystery affine map:



$a\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ $a\begin{bmatrix}1\\0\end{bmatrix}$ $a\begin{bmatrix}0\\0\end{bmatrix}$

2a) Find the formula for the mystery affine map. (15 points)

$$a\begin{pmatrix}x\\y\end{pmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}e\\f\end{bmatrix}$$

$$\begin{bmatrix}e\\f\end{bmatrix} = a\begin{pmatrix}0\\0\end{pmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$a\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} = col_1(A) + \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\text{so } col_1(A) = \begin{bmatrix}3\\-1\end{bmatrix} - \begin{bmatrix}1\\-2\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$a\begin{pmatrix}0\\1\end{pmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = col_2(A) + \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\text{so } col_2(A) = \begin{bmatrix}-1\\1\end{bmatrix}$$

$$\boxed{a\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}1\\-2\end{bmatrix}}$$

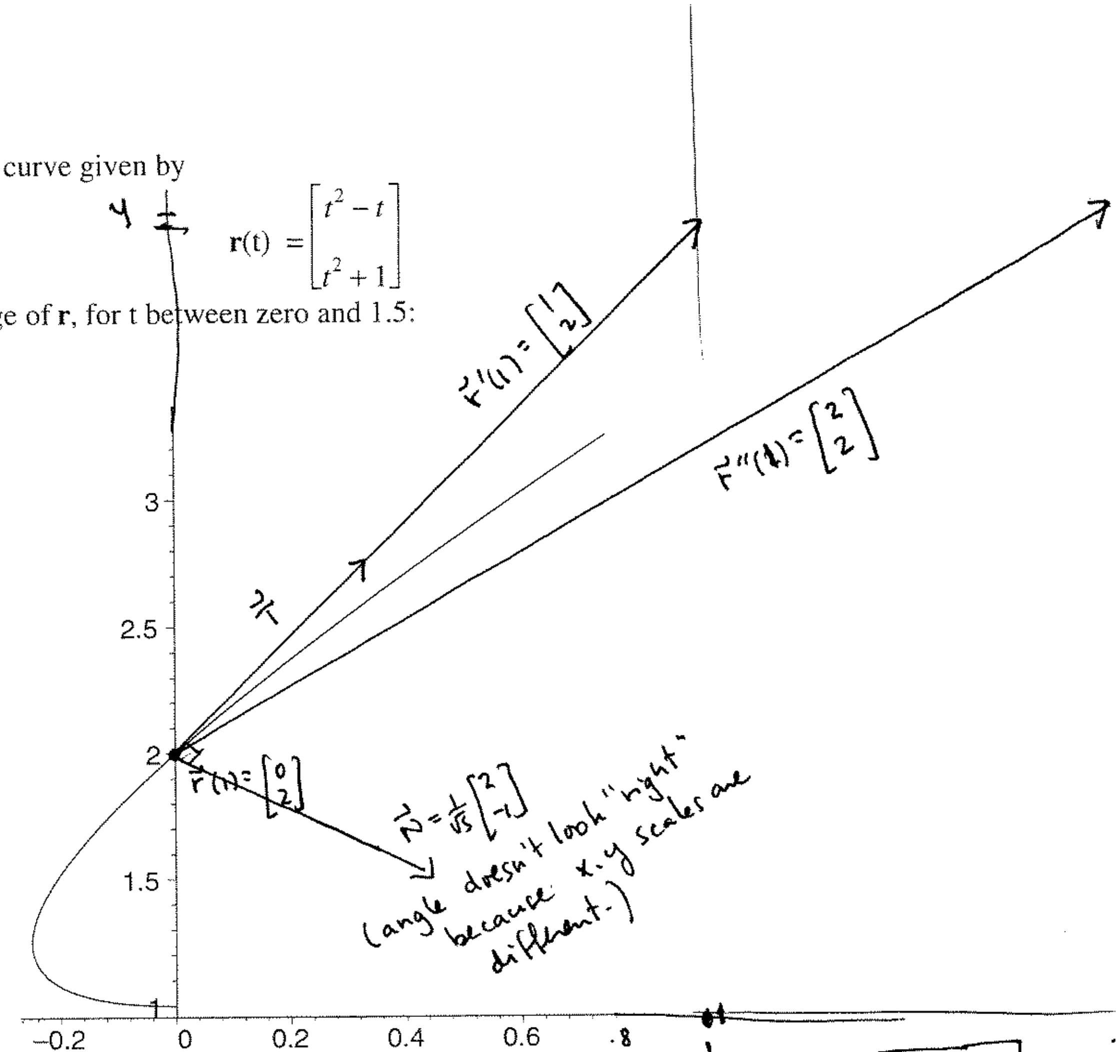
2b) What is the area of the image L-box? (5 points)

$$\text{area} = \left|\det\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\right| = 3$$

3) Consider the parametric curve given by

$$\mathbf{r}(t) = \begin{bmatrix} t^2 - t \\ t^2 + 1 \end{bmatrix}$$

Here is a sketch of the range of **r**, for t between zero and 1.5:



$= \sqrt{8t^2 - 4t + 1}$

3a) Compute $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. $\vec{r}'(t) = \begin{bmatrix} 2t-1 \\ 2t \end{bmatrix}$; $\vec{r}''(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $v = |r'(t)| = \sqrt{(2t-1)^2 + (2t)^2}$ ~~dt~~

(4 points)

3b) Write down a definite integral which yields the length of the parametric curve **r**(t), for t in the interval between zero and 2. Do not evaluate this integral. $\int_0^2 \sqrt{(2t-1)^2 + (2t)^2}\, dt$

(6 points)

3c) Label the point with position vector **r**(1) into the picture above. Add the vectors $\mathbf{r}'(1)$ and $\mathbf{r}''(1)$ to the picture, in the appropriate locations. $\vec{r}'(1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{r}''(1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

(5 points)

3d) Find the component of $\mathbf{r}''(1)$ in the direction of $\mathbf{r}'(1)$ using the dot product.

(5 points)

$$\text{comp}_{\vec{a}} \vec{b} = \vec{b} \cdot \left(\frac{\vec{a}}{|\vec{a}|}\right)$$

$$\text{comp}_{\vec{r}'(1)} \vec{r}''(1) = \frac{\vec{r}''(1) \cdot \vec{r}'(1)}{|\vec{r}'(1)|} = \frac{6}{\sqrt{5}}$$

3e) In class we derived the deomposition formula for acceleration, decomposing it into tangential and normal parts. The formula was

$$\mathbf{r}''(t) = \left(\frac{d}{dt}\mathrm{v}(t)\right)T + \kappa v^2 N$$

where T and N are the unit tangent and normal vectors, v is the speed, and κ is the curvature. Add the vectors T and N to your picture above. (Hint: you can find N easily because it is perpendicular to T and the curve lies in a plane.) $(t=1)$ $\vec{T} = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$, so $\vec{N} = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$ (4 points)

3f) Verify that the component you computed in (3d) agrees with $\frac{d}{dt}\mathrm{v}(t)$, when t=1.

(5 points)

3g) Use the formula from (3e), and your previous computations, to deduce the curvature of the parametric curve $\mathbf{r}(t)$, when t=1.

(6 points)

$v(t) = (8t^2-4t+1)^{1/2}$

$v'(t) = \frac{1}{2}(8t^2-4t+1)^{-1/2} \cdot (16t-4)$

$v'(1) = \frac{1}{2}(5)^{-1/2}(12) = \frac{6}{\sqrt{5}}$

and, comp$_{\vec{r}'(1)}$ $\vec{r}''(1) \doteq \frac{6}{\sqrt{5}}$ from page 1 ← agree!

$\vec{r}''(1) = v'(1)\vec{T} + \kappa v(1)^2 \vec{N}$

$\begin{bmatrix}2\\2\end{bmatrix} = \frac{6}{\sqrt{5}}\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix} + \kappa \cdot 5\,\vec{N}$

So $5\kappa\vec{N} = \begin{bmatrix}2\\2\end{bmatrix} - \frac{6}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}4/5\\-2/5\end{bmatrix} = \frac{2}{5}\begin{bmatrix}2\\-1\end{bmatrix}$

So (taking magnitudes)

$5\kappa = \frac{2}{5}\sqrt{5}$

$\boxed{\kappa = \frac{2}{(\sqrt{5})^3}}$