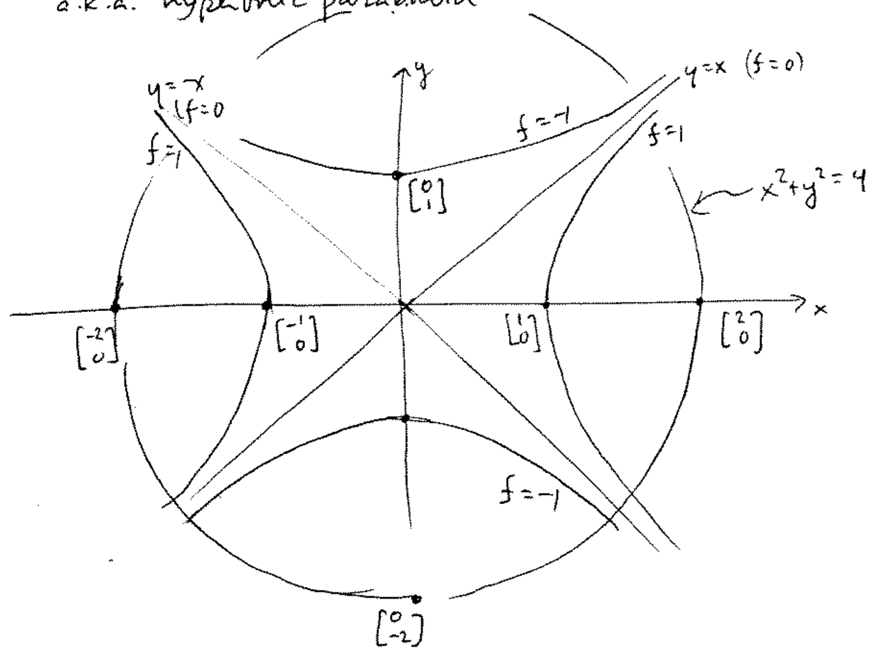
Math 2210-1
Practice Exam 2 Solutions
10/26/04

1 a) $z = x^2 - y^2$ describes a saddle surface
a.k.a. hyperbolic paraboloid

b).
c)



d) Maximize
$$f\begin{pmatrix}x\\y\end{pmatrix} = x^2 - y^2$$
subject to
$$g\begin{pmatrix}x\\y\end{pmatrix} = x^2 + y^2 = 4$$

$\nabla f = \lambda \nabla g$

$\langle 2x, -2y\rangle = \lambda\langle 2x, 2y\rangle$

$x = \lambda x$
$-y = \lambda y$

$x = 0$
$\Downarrow$
$y = \pm 2$
$\begin{bmatrix}0\\2\end{bmatrix}, \begin{bmatrix}0\\-2\end{bmatrix}$
$f\left(\begin{bmatrix}0\\2\end{bmatrix}\right) = f\left(\begin{bmatrix}0\\-2\end{bmatrix}\right) = -4$
$\uparrow$
min value.

$x \neq 0$
$\Downarrow$
$\lambda = 1$
$\Downarrow$
$y = 0$
$\Downarrow$
$x = \pm 2$
$f\left(\begin{bmatrix}\pm 2\\0\end{bmatrix}\right) = 4$
$\uparrow$
max value

the level curves of f are hyperbolas.
When $f > 0$ these hyperbolas open along the x-axis, and the $f = 4$ hyperbola, with vertices at $\begin{bmatrix}\pm 2\\0\end{bmatrix}$ is the highest value level curve to intersect the circle $x^2+y^2=4$.

When $f < 0$ these hyperbolas open along the y-axis, and the lowest value level curve to intersect the circle is $f = -4$, with vertices at $\begin{bmatrix}0\\\pm 2\end{bmatrix}$.

2. $z = f(x,y) = x^2 - y^2$

$\nabla f = \langle 2x, -2y \rangle$

$\nabla f(1,1) = \langle 2,-2 \rangle$. $\quad f(1,1) = 0$

(a) tang plane $z = 0 + 2(x-1) - 2(y-1)$

or $\boxed{z - 2x + 2y = 0}$

(b) $f(x+\Delta x, y+\Delta y) \cong f(x,y) + \underbrace{df}_{f_x \Delta x + f_y \Delta y}$.

$x=1 \quad y=1$
$\Delta x = .1 \quad \Delta y = -.1$

$f(1.1, .9) \approx f(1,1) + 2(.1) - (2)(-.1)$

$\approx \underline{.4}$

$f(1.1, .9) = (1.1)^2 - (.9)^2 = \begin{array}{r} 1.21 \\ -\ .81 \\ \hline .4 \end{array} = .4$

(c) using tangent plane,

$z = 0 + 2(.1) - 2(-.1)$ gives same value as df approximation.
$= .4$ (this is always the case)

(d) $\vec{r}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

$\vec{r}'(0) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\quad$ @ $t=0$, $\langle 2,-2 \rangle \cdot \langle 5,4 \rangle = 10 - 8 = \boxed{2}$

(e) in the direction of $\nabla f = \langle 2,-2 \rangle$

i.e. $\vec{u} = \frac{1}{\sqrt{2}} \langle 1,-1 \rangle$

3. $\vec{F}: \mathbb{R}^n \to \mathbb{R}^m$ is

(a) continuous at $\vec{x}$ iff $\lim_{\vec{z} \to \vec{x}} \vec{F}(\vec{z}) = \vec{F}(\vec{x})$ $\quad$ (or $\lim_{\vec{h} \to 0} \vec{F}(\vec{x}+\vec{h}) = \vec{F}(\vec{x})$).

(d) diff'ble at $\vec{x}$ iff $\exists$ a matrix $[\vec{F}'(\vec{x})]$ (entry$_{ij} = \frac{\partial F_i}{\partial x_j}(\vec{x})$) s.t.

$\vec{F}(\vec{x}+\vec{h}) = \vec{F}(\vec{x}) + [\vec{F}'(\vec{x})]\vec{h}$ + error

where $\lim_{\vec{h} \to 0} \frac{|\text{error}|}{|\vec{h}|} = 0$.

4. $\vec{F}\begin{bmatrix} \rho \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \rho \sin\phi \cos\theta \\ \rho \sin\phi \sin\theta \\ \rho \cos\phi \end{bmatrix}$

$$[F'] = \begin{array}{cc} & \begin{array}{ccc} \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \end{array} \\ & \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix} \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} \end{array}$$