limits and continuity: We talked about this for parametric curves $F : [a,b] \rightarrow \mathbb{R}^n$
General case is analogous

$\mathbb{F} : \mathbb{D} \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$

$F([x_1, x_2, \ldots, x_n]) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{bmatrix}$

Remark: Since two points in $\mathbb{R}^m$ or $\mathbb{R}^n$ are close if and only if all of their corresponding components are close, we can do easy limit computations using limit facts from Calculus.
Example
\[ \lim_{[x, y] \to [1, 2]} \begin{bmatrix} 4x^2 + \sin(xy) \\ e^y \end{bmatrix} = \begin{bmatrix} \text{limits} \\ \text{limits} \end{bmatrix} \]

used:
- limit of sum is sum of limits
- limit of product is product of limits
- you can pass limits through continuous real valued functions.

\textbf{Definition:} \( \mathbf{F}: \mathbb{R}^n \to \mathbb{R}^m \) is continuous at \( x_0 \) iff \( \lim_{x \to x_0} \mathbf{F}(x) = \mathbf{F}(x_0) \).

(just like calc def for scalar functions.)

In section 13.3 we focus on
\( f: D \to \mathbb{R} \)
\( \mathbb{R}^3 \)

and often use the graph
of \( f \) in \( \mathbb{R}^3 \) to visualize limit questions.

On Maple handout:

1) \( \lim_{(x,y) \to (0,0)} \sin(x^2+y^2) = \)

2) \( \lim_{(x,y) \to (0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \)
3) \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \)

* use polar coords

4) \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2+y^2} = \)

* use polar coords
* use approach along \( y = mx \).
In the case that \( f : \mathbb{R}^2 \to \mathbb{R} \), it is convenient to study limit and continuity questions using the graph 
\[ z = f(x,y) \] 
of \( f \). We will discuss several interesting limits as \((x,y) \to (0,0)\). Some of these are examples or homework problems from the text section 13.3, or modifications of them.

1) Find the limit as \((x,y) \to (0,0)\) of
\[ f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \]
Is your answer consistent with the graph of \( f \)? Is \( f \) continuous at the origin?

- \( \text{plot3d}(f(x,y), x=-2..2, y=-2..2, scaling=constrained, color=white, axes=boxed, title='can you explain this graph?'); \)

**Remark:** Maple plots the graph of a function of 2 variables parametrically. For example, in the picture above, it takes the square \( x=-2..2, y=-2..2 \) and cross-hatches it with a 20 by 20 grid (you can change this default option). Then it sends \((x,y) \to (x,y,f(x,y))\), and traces where the grid goes. You can recover the domain grid by looking at the plot from above.

2) Can you find the limit as \((x,y) \to (0,0)\) of
\[ g(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \]
Is your answer consistent with the graph of \( g \)? Could you define \( g(0,0) \) to make \( g(x,y) \) continuous at the origin?

- \( \text{plot3d}(g(x,y), x=-2..2, y=-2..2, scaling=constrained, orientation=[42,75], color=white, axes=boxed, title='correct limit?'); \)

```maple
> restart;
with(plots):

> f := (x,y) -> sin(x^2+y^2)/(x^2+y^2):

> plot3d(f(x,y), x=-2..2, y=-2..2, scaling=constrained, color=white, axes=boxed, title='can you explain this graph?');

> g := (x,y) -> sin(x^2+y^2)/(x^2+y^2):

> plot3d(g(x,y), x=-2..2, y=-2..2, scaling=constrained, orientation=[42,75], color=white, axes=boxed, title='correct limit?');
```
3) Here is a more interesting limit we will try to analyze, perhaps using polar coordinates:

\[ h(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} \]

\[ k(x, y) = \frac{xy}{x^2 + y^2} \]

4) How about this one:

\[ k(x, y) = \frac{xy}{x^2 + y^2} \]
If we use polar coordinates we can parameterize this surface with $r$ and $\theta$ (we should work this parameterization out in class):

```matlab
> plot3d([r*cos(theta), r*sin(theta), sin(2*theta)/2], r=0..1, theta=0..2*Pi, grid=[30,30], axes=boxed, scaling=constrained, color=white, orientation=[126,51], title='another view');
```

5) You might think from the previous two examples that if you just check along radial lines you can always figure out whether a limit as $(x,y) \to (0,0)$ exists, or that using polar coordinate will solve every such problem. Although such reasoning often suffices, here's a cool example that shows you would be wrong. The first function we graph has value 1 when $y < x^2$ or $y > 2x^2$, and is zero for $x^2 < y < 2x^2$. Every radial limit of $f$ as you approach the origin (along rays) is zero, and yet the limit of $f(x,y)$ as $(x,y) \to (0,0)$ is not defined!

```matlab
> plot3d(Heaviside((y-x^2)^2/(y-x^2)), x=-1..1, y=-1..1, grid=[40,40], axes=boxed, contours=10, title='parabolic minima I');
```
The first parabolic narrows function is discontinuous along the parabolas $y=x^2$ and $y=2^x x^2$. We can create a slight slope where the cliffs were to make it continuous everywhere except at the origin:

```matlab
> plot3d(1-abs(y/x^2-2)),
Heaviside(3*x^2-y)*Heaviside(y-x^2),
x=-1..1,y=-1..1,grid=[40,40],
contours=20,
axes=boxed3,
title='parabolic narrows II');
```

You have a homework problem like this example, but with a "simpler" formula, on which you should study behavior along parabolas to deduce a limit does not exist.