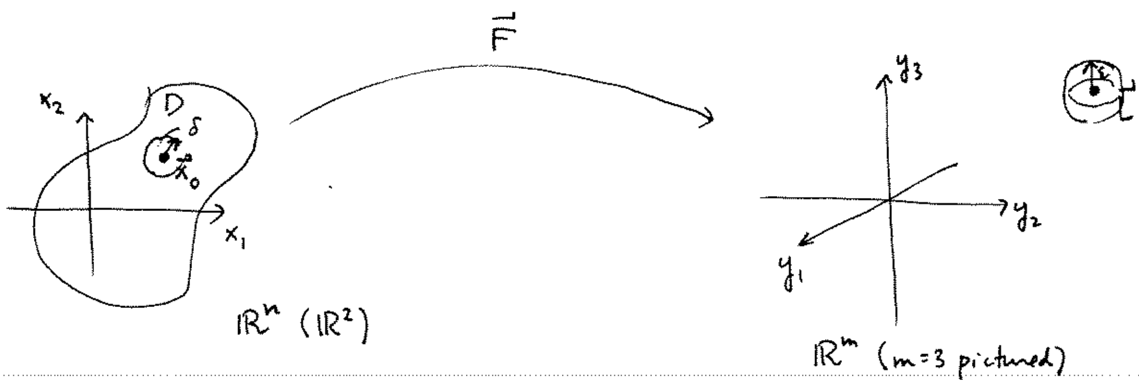
Math 2210-1
Wed 10/6

Homework for Wed 10/13
13.2 (3, 6, 13, 35, 37, 41, 43, 53-58)
13.3 (4, 15, 21, 29, 41, 42, 51, 54)

limits and continuity : We talked about this for parametric curves $\vec{r} : [a,b] \to \mathbb{R}^n$
General case is analogous

$$\vec{F} : D \longrightarrow \mathbb{R}^m, \quad D \subset \mathbb{R}^n$$

$$\vec{F}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$



$$\lim_{\vec{x} \to \vec{x}_0} \vec{F}(\vec{x}) = \vec{L} \quad \text{means (precisely)}$$

$$\forall \varepsilon > 0 \ \exists \ \delta > 0 \ \text{s.t.} \quad 0 < |\vec{x} - \vec{x}_0| < \delta \implies |\vec{F}(\vec{x}) - \vec{L}| < \varepsilon$$

$\forall$ "for all"; $\exists$ "there exists"; s.t. "such that"; $\implies$ "implies"

intuitively: if you take $\vec{x}$ close enough to $\vec{x}_0$ ($\& \neq \vec{x}_0$)
then $\vec{F}(\vec{x})$ can be made as close as you want to $\vec{L}$

<u>Remark</u> : Since two points in $\mathbb{R}^n$ or $\mathbb{R}^m$ are close if and only if ~~each~~ all of their corresponding components are close, we can do easy limit computations using limit facts from Calculus.

example

$$\lim_{\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \begin{bmatrix} 4x^2 + \sin(xy) \\ e^{xy} \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

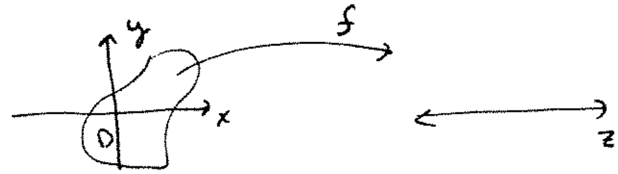
used: limit of sum is sum of limits
limit of product is product of limits
you can pass limits through continuous real valued functions.

Definition: $\vec{F}: D \to \mathbb{R}^m$, $D \subset \mathbb{R}^n$ is continuous at $\vec{x}_0$ iff $\lim_{\vec{x} \to \vec{x}_0} \vec{F}(\vec{x}) = \vec{F}(\vec{x}_0)$.

(just like calc def for scalar functions.

In section 13.3 we focus on

$f: D \to \mathbb{R}$, $D \subset \mathbb{R}^2$



and often use the graph of f in $\mathbb{R}^3$ to visualize limit questions.

On Maple handout:

1) $\lim_{(x,y) \to (0,0)} \sin(x^2+y^2) =$

2) $\lim_{(x,y) \to (0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} =$

3) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} =$

- use polar coords

4) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} =$

- use polar coords
- use approach along $y=mx$.

5) $\lim_{(x,y)\to(0,0)} f(x,y)$

$$f(x,y) = \begin{cases} 1 & y > 2x^2 \text{ or } y < x^2 \\ 0 & \text{if } x^2 < y < 2x^2 \end{cases}$$

Math 2210-1
October 6, 2004
Visualizing limits

In the case that f: R^2->R, it is convenient to study limit and continuity questions using the graph $z = f(x, y)$ of f. We will discuss several interesting limits as (x,y)->(0,0). Some of these are examples or homework problems from the text section 13.3, or modifications of them.

```
> restart:
  with(plots):
```

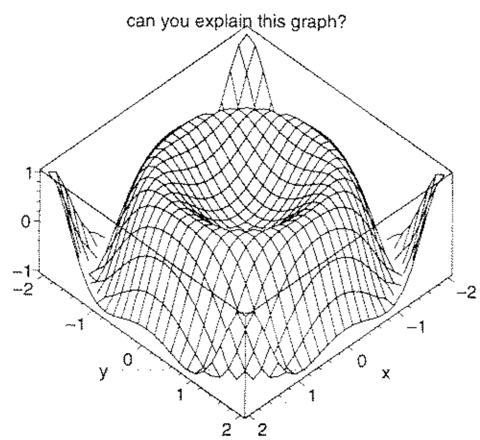
1) Find the limit as (x,y)->(0,0) of

```
> f:=(x,y)->sin(x^2+y^2);
```

$$f := (x, y) \to \sin(x^2 + y^2)$$

Is your answer consistent with the graph of f? Is f continuous at the origin?

```
> plot3d(f(x,y),
  x=-2..2,y=-2..2,scaling=constrained,
  color=white,
  axes=boxed,
  title=`can you explain this graph?`);
```



Remark: Maple plots the graph of a function of 2 variables parametrically. For example, in the picture above, it takes the square x=-2..2, y=-2..2 and cross-hatches it with a 20 by 20 grid (you can change this default option). Then it sends (x,y) ---->(x,y,f(x,y)), and traces where the grid goes. You can recover the domain grid by looking at the plot from above.

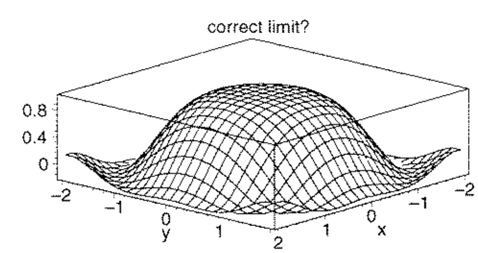
2) Can you find the limit as (x,y)->(0,0) of

```
> g:=(x,y)->sin(x^2+y^2)/(x^2+y^2);
```

$$g := (x, y) \to \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

Is your answer consistent with the graph of g? Could you define g(0,0) to make g(x,y) continuous at the origin?

```
> plot3d(g(x,y),
  x=-2..2,y=-2..2,scaling=constrained,
  orientation=[42,75],
  color=white,
  axes=boxed,
  title=`correct limit?`);
```

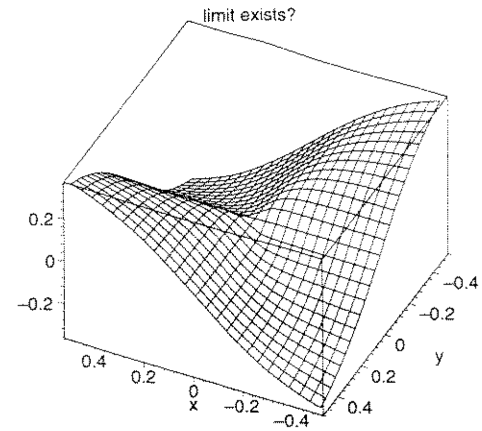


3) Here is a more interesting limit we will try to analyze, perhaps using polar coordinates:

```
> h:=(x,y)->x*y/(sqrt(x^2+y^2));
```

$$h := (x, y) \rightarrow \frac{x\,y}{\sqrt{x^2 + y^2}}$$

```
> plot3d(h(x,y),
  x=-.5..(.5),y=-.5..(.5),scaling=constrained,
  orientation=[116,51],
  color=white,
  axes=boxed,
  title=`limit exists?`);
```

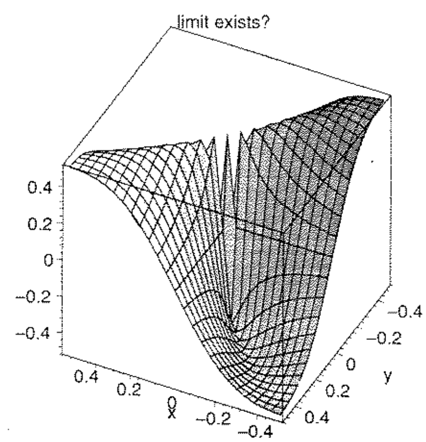


4) How about this one:

```
> k:=(x,y)->x*y/(x^2+y^2);
```

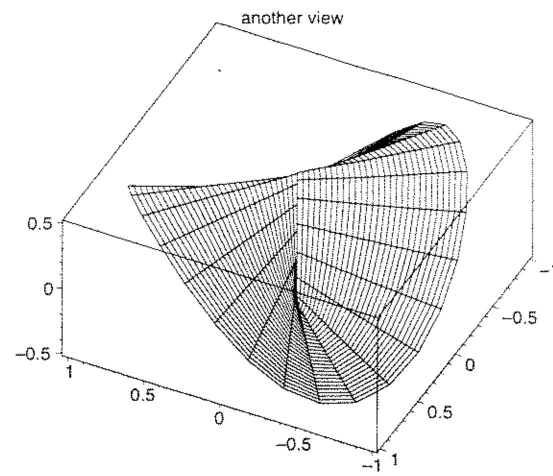
$$k := (x, y) \rightarrow \frac{x\,y}{x^2 + y^2}$$

```
> plot3d(k(x,y),
  x=-.5..(.5),y=-.5..(.5),scaling=constrained,
  orientation=[116,51],
  contours=20,
  axes=boxed,
  title=`limit exists?`);
```



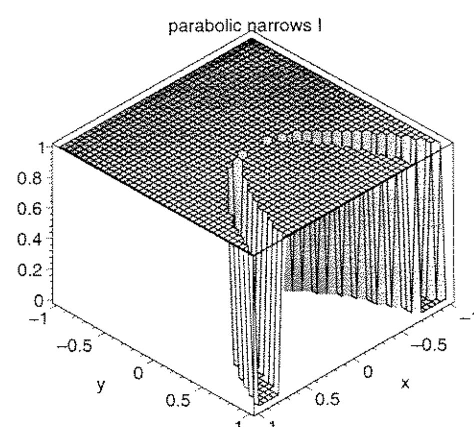
If we use polar coordinates we can parameterize this surface with "r" and "theta" (we should work this parameterization out in class):


```
> plot3d([r*cos(theta),r*sin(theta),sin(2*theta)/2],
  r=0..1,theta=0..2*Pi,
  grid=[30,30],axes=boxed,
  scaling=constrained,
  color=white,
  orientation=[116,51],
  title=`another view`);
```



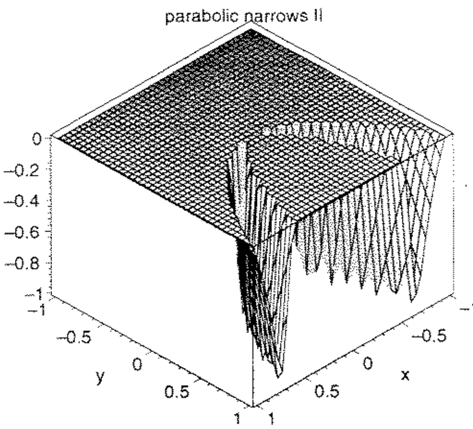
5) You might think from the previous two examples that if you just check along radial lines you can always figure out whether a limit as (x,y)->(0,0) exists, or that using polar coordinate will solve every such problem. Although such reasoning often suffices, here's a cool example that shows you would be wrong. The first function we graph has value 1 when y<x^2 or y>2*x^2, and is zero for x^2<y<2*x^2. Every radial limit of f as you approach the origin (along rays) is zero, and yet the limit of f(x,y) as (x,y)-->0 is not defined!

```
> plot3d(Heaviside((y-2*x^2)*(y-x^2)),
  x=-1..1,y=-1..1,grid=[40,40],
  axes=boxed,
  contours=20,
  title=`parabolic narrows I`);
```



The first parabolic narrows function is discontinuous along the parabolas y=x^2 and y=2*x^2. We can create a slight slope where the cliffs were to make it continuous everywhere except at the origin:

```
> plot3d(-(1-abs(y/x^2-2))*
  Heaviside(3*x^2-y)*Heaviside(y-x^2),
  x=-1..1,y=-1..1,grid=[40,40],
  contours=20,
  axes=boxed,
  title='parabolic narrows II');
```



You have a homework problem like this example, but with a "simpler" formula, on which you should study behavior along parabolas to deduce a limit does not exist.

```
>
>
>
```