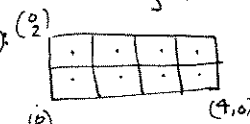
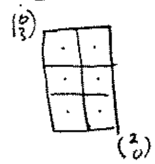
Math 2210-1
Fri Oct 29.

Multiple Integration
§14.1 Double integrals.

HW for Wed Nov. 3

14.1 11, 12 ← also, in 11, 12 do a Riemann sum approximation for each integral, using unit length squares and evaluating f at the midpoints.

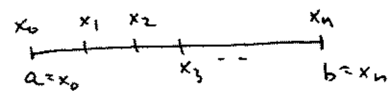
11. 

12. 

14.1 cont'd 27, 31, 32
14.2 3, 7, 8, 15, 18, 23, 25, 29, 31

Remember 1210 definite integral:

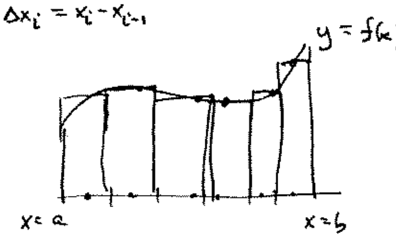
$$\int_a^b f(x)\,dx = \lim_{\max(\Delta x_i)\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$



i^{th} interval

$\Delta x_i = x_i - x_{i-1}$

we motivated this definition by thinking about area under a graph $y = f(x)$, but the definite integral had lots of other applications:



- area between graphs
- total mass, if $f(x)$ = mass/length
- work, if $f(x)$ = force
- curve length
- volumes (e.g. of revolution)
- surface areas of revolution
- centers of mass.

- If f is continuous on $[a,b]$, then the definite integral exists.
- Fundamental Thm of Calc says you can compute
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
where F is any antideriv. of f.

Double integrals over rectangles

Let $R = [a,b] \times [c,d] = \{ (x,y) \in \mathbb{R}^2 \text{ s.t. } a \le x \le b,\ c \le y \le d \}$.
be a coordinate rectangle.

Let $z = f(x,y)$ a function with domain R.

Partion P of rectangle.

$a = x_0 < x_1 < \cdots < x_m = b$

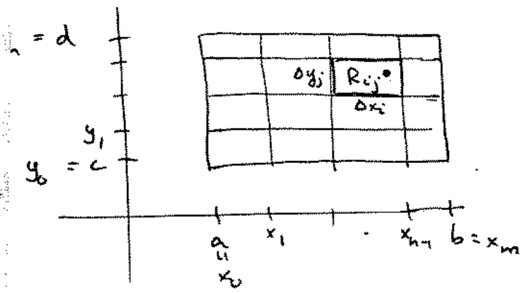
$c = y_0 < y_1 < \cdots < y_n = d$.

yields subrectangles

$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, with area $\Delta A_{ij} = (\Delta x_i)(\Delta y_j)$

pick points

(x_i^*, y_j^*) in R_{ij}



$J_j = [y_{j-1}, y_j)$

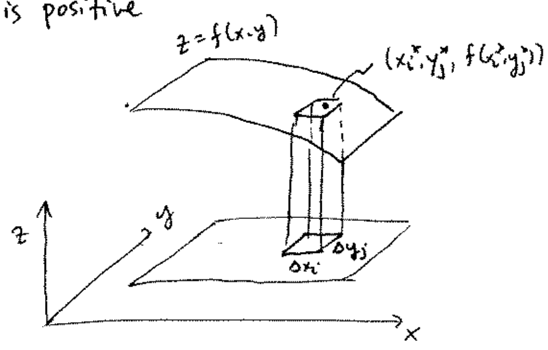
$I_i = [x_{i-1}, x_i]$

then, for this partition P, and function f, get

Riemann Sum

$$\sum_{\substack{i=1 \cdots m \\ j=1 \cdots n}} f(x_i^*, y_j^*) \underbrace{\Delta A_{ij}}_{(\Delta x_i)(\Delta y_j)}.$$

this Riemann sum can be thought of an approximation of the volume between $z = f(x,y)$ & the x-y plane, if f is positive

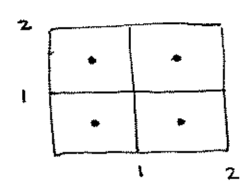


<u>Example</u>. (to do!)

$f(x,y) = x + y$

$R = [0,2] \times [0,2]$.

P:



Find the Riemann sum using midpoints ($\bullet$'s)

<u>Theorem</u> If f is continuous on the rectangle R, then

$$\lim_{\max \Delta x_i, \Delta y_j \to 0} (\text{Riemann sum of } f) := \iint_R f(x,y)\, dA$$

exists. (Called the double integral of f over the region R).

Luckily, it is easy to compute ...

<u>Theorem</u>: Double integrals as iterated single integrals, (for f continuous on rectangle $[a,b]\times[c,d]$.)

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\underbrace{\int_a^b f(x,y)\,dx}_{\text{treat } y \text{ as a constant and integrate with respect to } x.} \right) dy$$

idea of why: consider Riemann sum

$$\sum_{\substack{i=1\cdots m\\ j=1\cdots n}} f(x_i^*, y_j^*)(\Delta x_i)\,\Delta y_j$$

if we let $\max \Delta x_i \to 0$, keeping $y_j^*, \Delta y_j$ fixed this should converge to

$$\sum_{j=1\cdots n} \int_a^b f(x, y_j^*)(\Delta y_j)\,dx = \sum_{j=1}^{n} \left(\int_a^b f(x, y_j^*)\,dx \right) \Delta y_j$$

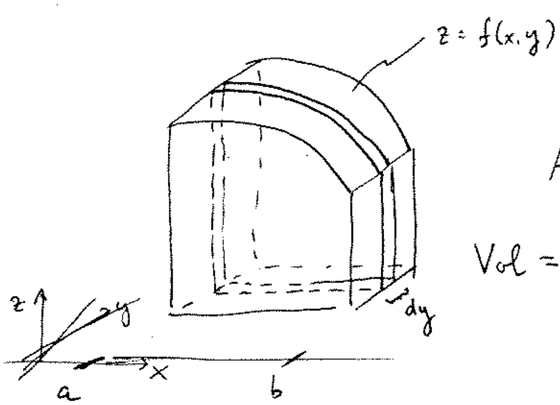
which is a Riemann sum for the function $G(y) = \int_a^b f(x,y)\,dx$, $c \le y \le d$

So, as $\Delta y_j \to 0$, this Riemann sum should converge to

$$\int_c^d G(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

(get reverse iteration analogously).

Another way to "see" this is to recall how you found volumes by slicing, in Calculus.



$$A(y) = \text{area when } y\text{-coord is } y = \int_a^b f(x,y)\,dx$$

$$\text{Vol} = \iint f(x,y)\,dA = \int_c^d \underbrace{A(y)\,dy}_{dV} = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy.$$

example.

$\mathcal{R} = [0,2]\times[0,2]$

$f(x,y) = x+y.$

$$\iint_{\mathcal{R}} (x+y)\,dA = \underset{\substack{\uparrow \\ x \text{ limits}}}{\int_0^2} \left(\int_0^2 x+y\,dy\right) dx = \int_0^2 \left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^2 2x+2\,dx$$

$$= x^2+2x\Big]_0^2 = 8$$

how does this compare
to the Riemann sum on page 2?
Accident?

Example $\mathcal{R} = [0,\pi/2]\times[0,1]$

$f(x,y) = e^y + \cos x$

compute $\iint_{\mathcal{R}} f(x,y)\,dA$ as iterated integrals, in both orders.